Formal concept analysis (FCA)

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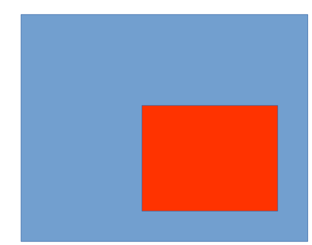
Relation

- binary relation on a set A is a collection of ordered pairs of elements of A.
- a subset of the Cartesian product $A^2 = A \times A$
- $R \subseteq A \times A$
- e.g. “divides”

Order theory

- Order
  - binary relation
  - <, >, ≤, ≥
  - e.g. numbers: 1 < 2 < 3 < …
  - Total order: for each pair a ≤ b or b ≥ a
  - Partial order
    - rectangles
      - A rectangle is “larger” if it covers the “smaller”
      - Not all pairs are either “larger” or “smaller”

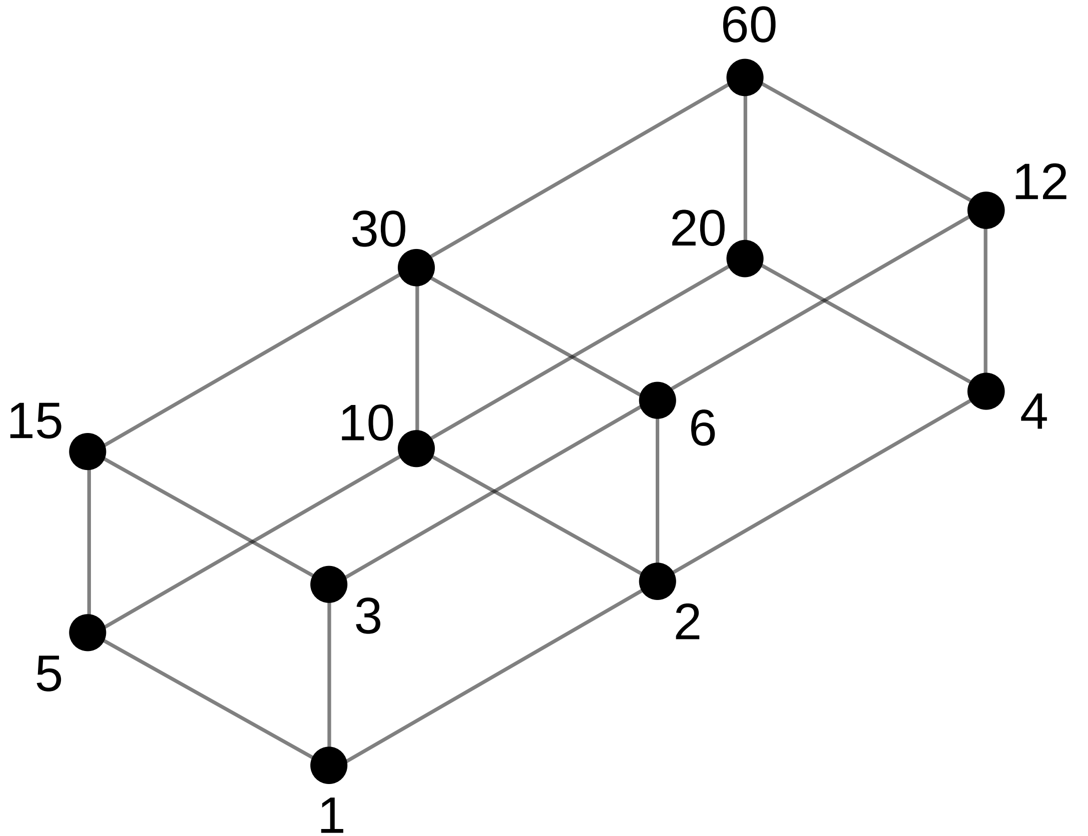


Order

- Then $\leq$ is a partial order if it is reflexive, antisymmetric, and transitive, i.e., for all a, b and c in P, we have that:

- $a \leq a$ *(reflexivity)*
- if $a \leq b$ and $b \leq a$ then $a = b$ *(antisymmetry)*
- if $a \leq b$ and $b \leq c$ then $a \leq c$ *(transitivity)*.

Hasse diagram



Chains and anti-chains

- total order: chain
- No comparable pairs: anti-chain

Formal Concept Analysis

- formal context: $K := (G, M, I)$
- formal objects: G
- formal attributes: M
- binary relation $I \subseteq G \times M$

Neuron example

- members of G are visual stimuli
- members of M are the neurons.
- if neuron m ∈ M responds when stimulus g ∈ G is presented, (g, m) ∈ I
- cross table:

Table 1 *Left*: a simple example context,

	n1	n2	n3
monkeyFace	×	×	
monkeyHand		×	
humanFace	×		
spider			×

Formal concept

The prime operator for subsets $A \subseteq G$ is defined as $A' = \{m \in M | \forall g \in A : gIm\}$, i.e., A' is the set of all attributes shared by the objects in A. Likewise, for $B \subseteq M$, B' is defined as $B' = \{g \in G | \forall m \in B : gIm\}$, i.e., B' is the set of all objects having all attributes in B.

Definition 1 ([9]) A **formal concept** of the context K is a pair (A, B) with $A \subseteq G$, $B \subseteq M$ such that $A' = B$ and $B' = A$. A is called the **extent** and B is the **intent** of the concept (A, B). $\mathbb{B}(K)$ denotes the set of all concepts of the context K.

In other words, given the relation I, (A, B) is a concept if A determines B and vice versa. A and B are sometimes called *closed* subsets of G and M with respect to I. Table 1, right, lists all concepts of the context in Table 1, left. One can visualise

Table 1 *Left*: a simple example context, represented as a cross-table

	n1	n2	n3
monkeyFace	×	×	
monkeyHand		×	
humanFace	×		
spider			×

concept	extent (stimuli)	intent (neurons)
0	*ALL*	*NONE*
1	spider	n3
2	humanFace monkeyFace	n1
3	monkeyFace monkeyHand	n2
4	monkeyFace	n1 n2
5	*NONE*	*ALL*

Concept lattice

Definition 2 [9] If (A_1, B_1) and (A_2, B_2) are concepts of a context, (A_1, B_1) is a **subconcept** of (A_2, B_2) if $A_1 \subseteq A_2$ (which is equivalent to $B_1 \supseteq B_2$). In this case, (A_2, B_2) is a **superconcept** of (A_1, B_1) and we write $(A_1, B_1) \leq (A_2, B_2)$. The relation $\leq$ is called the **order** of the concepts.

Table 1 *Left*: a simple example context, represented as a cross-table

	n1	n2	n3
monkeyFace	×	×	
monkeyHand		×	
humanFace	×		
spider			×

concept	extent (stimuli)	intent (neurons)
0	*ALL*	*NONE*
1	spider	n3
2	humanFace monkeyFace	n1
3	monkeyFace monkeyHand	n2
4	monkeyFace	n1 n2
5	*NONE*	*ALL*

Concept lattice

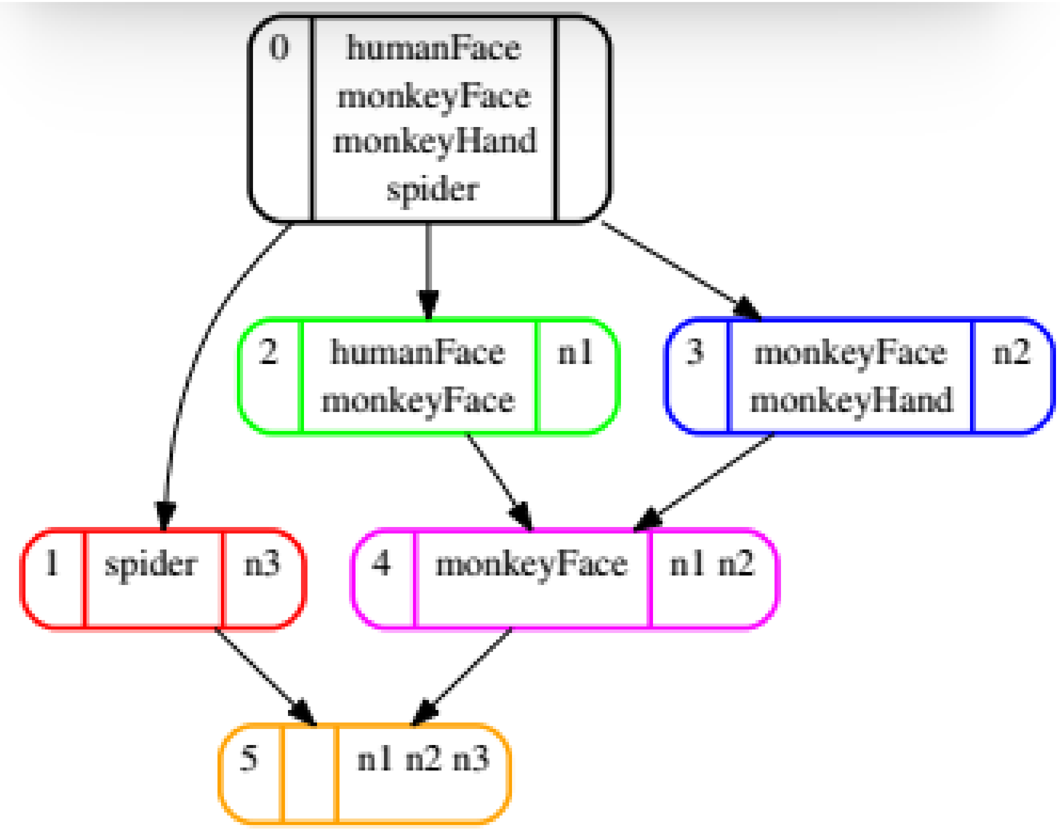


Table 1 *Left*: a simple example context, represented as a cross-table

	n1	n2	n3
monkeyFace	×	×	
monkeyHand		×	
humanFace	×		
spider			×

concept	extent (stimuli)	intent (neurons)
0	*ALL*	*NONE*
1	spider	n3
2	humanFace monkeyFace	n1
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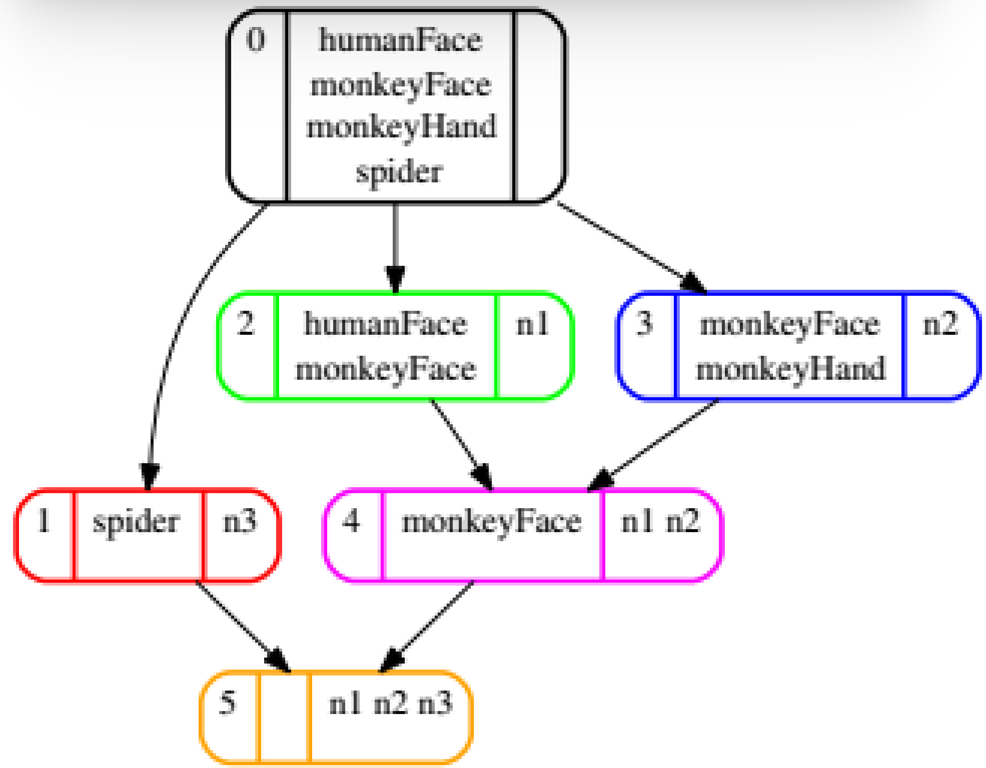
4	monkeyFace	n1 n2
5	*NONE*	*ALL*

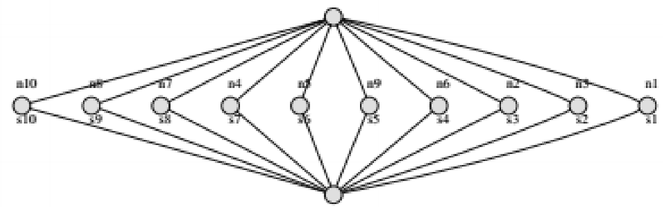


FCA algorithms

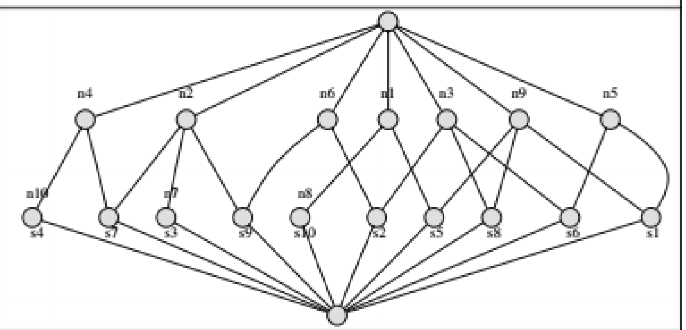
- Efficient algorithms exist for calculating concept lattice (a DAG – directed acyclic graph) from a cross table
- “you give me the attributes, I give you the concept graph”
- And vice versa: calculate attributes from a concept graph
- “you give me a concept graph and I encode it for you in binary sets/vectors”
- Direct mind – brain connection

Sparcity and concept lattices

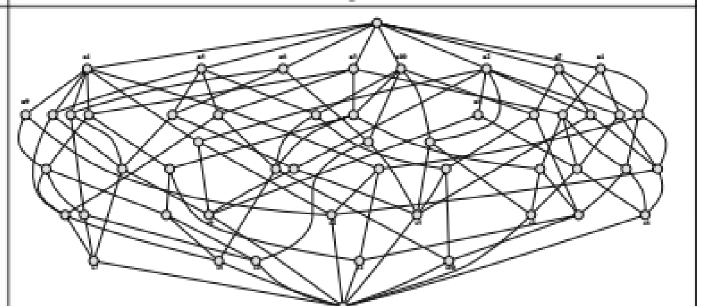
	n1	n2	n3	n4	n5	n6	n7	n8	n9	n10
s1	×									
s2			×							
s3		×								
s4						×				
s5									×	
s6					×					
s7				×						
s8							×			
s9								×		
s10										×



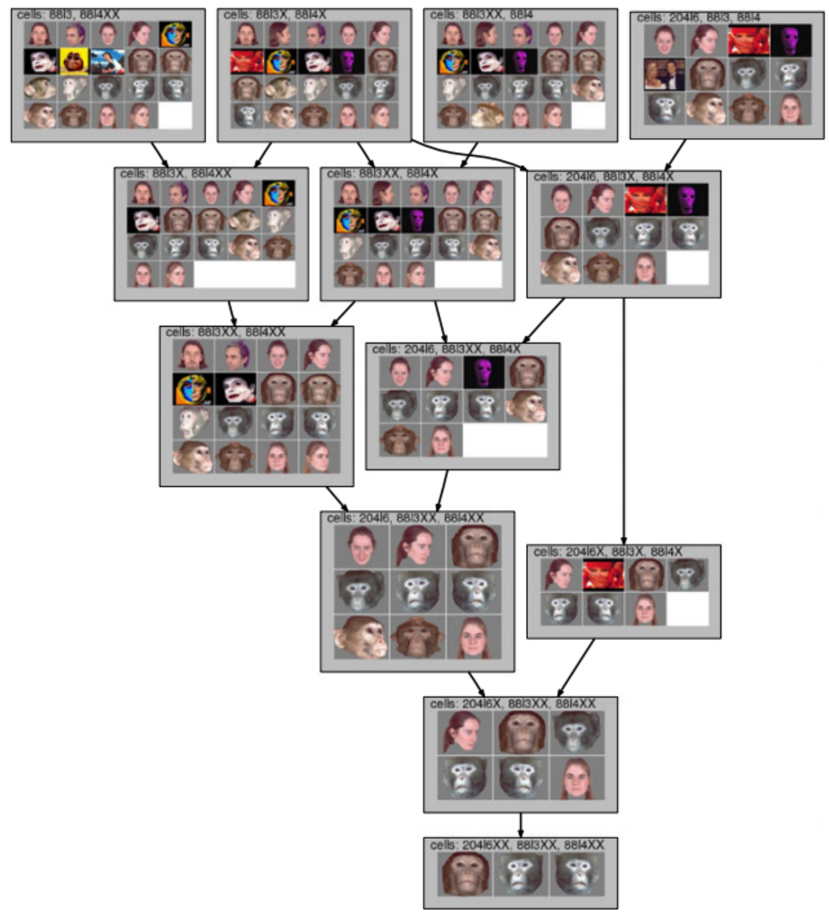
	n1	n2	n3	n4	n5	n6	n7	n8	n9	n10
s1					×				×	
s2			×			×				
s3		×					×			
s4				×						×
s5	×								×	
s6			×		×					
s7		×		×						
s8			×						×	
s9		×				×				
s10	×							×		



	n1	n2	n3	n4	n5	n6	n7	n8	n9	n10
s1	×		×	×			×			×
s2	×	×		×			×		×	
s3		×	×	×				×		×
s4	×		×			×			×	×
s5		×			×	×	×			×
s6		×	×	×			×	×		
s7	×		×		×	×			×	
s8	×		×		×				×	×
s9	×	×			×	×		×		
s10			×	×	×		×			×



Monkey high-level visual cortex single-cell FCA



Further research

- Disadvantage: any object with new attribute set introduces a concept.
  - Bad for large, noisy data sets
- Probabilistic extension:
  - Probabilistic Formal Concept Analysis
  - Introduce only “good” (significant) concepts
- Categorisation
- Associative, content-addressable memory

References

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