The method of test functions

for linear, inhomogeneous, constant coefficient case

Gábor Borbély

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1 Inaccurate problem statement

In this paper we consider inhomogeneous, linear differential equations in one dimension with constant coefficients. Namely

$$y^{(n)}(x) + a_{n-1} \cdot y^{(n-1)}(x) + \ldots + a_1 \cdot y'(x) + a_0 \cdot y(x) = f(x) \qquad (1)$$

where $a_0, a_1, \ldots a_{n-1} \in \mathbb{R}$ and $f : \mathbb{R} \mapsto \mathbb{R}$ is given and $y : \mathbb{R} \mapsto \mathbb{R}$ is sought.

We can solve the homogeneous case via the characteristic equation

$$\lambda^n + a_{n-1} \cdot \lambda^{n-1} + \ldots + a_1\lambda + a_0 = 0 \qquad (2)$$

and we have

$$y_h^{(n)}(x) + a_{n-1} \cdot y_h^{(n-1)}(x) + \ldots + a_1 \cdot y_h'(x) + a_0 \cdot y_h(x) \equiv 0 \qquad (3)$$

where y_h is a quasi-polynomial, containing (real) exponential, trigonometrial and polynomial functions (recall the matrix exponential).

The next step is to find a particular solution of the inhomogeneous equation. At this point we restrict ourselves to the case, when $f(x)$ has a special form, specified later.

Example 1.1.

$$y''(x) + y(x) = x \qquad (4)$$

The characteristic equation is $\lambda^2 + 1 = 0 \Rightarrow \lambda_{1,2} = \pm i$. *Therefore*

$$y_h(x) = c_1 \cdot \sin(x) + c_2 \cdot \cos(x) \qquad (5)$$

We search the particular solution in the following (linear) form

$$y_{ip}(x) = Ax + B \qquad (6)$$

This y_{ip} *is the* test function*, we assume that there is an element in* $\{Ax + B | A, B \in \mathbb{R}\} \subseteq C^\infty(\mathbb{R})$ *such that it solves (4). If we find one, the the general solution is* $c_1 \cdot \sin(x) + c_2 \cdot \cos(x) + Ax + B$ *with the above determined* A *and* B*.*

We will show a linear-algebraic formulation of the above techniques and we will propose some accurate questions later. The solution of the homogeneous case is simpler, however non-trivial, what about the particular solution? What test function to choose? If one has already proposed a test function (a supposed form of the solution), how to determine the coefficients? Are the coefficients uniquely determined?

2 Linear algebra formalism

Definition 2.1. *Let $\mathcal{F} \subseteq C^\infty(\mathbb{R})$ be a subspace in the space of smooth functions. We shall call $\mathcal{F}$ d-closed iff it is closed under derivation. Precisely $f \in \mathcal{F} \Rightarrow f' \in \mathcal{F}$.*

Claim 2.2. *If $\mathcal{F} \subseteq C^\infty(\mathbb{R})$ is d-closed then*

$$\operatorname{span}(\mathcal{F}) = \left\{ \sum_{i=1}^{n} c_i \cdot f_i \,\middle|\, n \in \mathbb{N}, c_i \in \mathbb{R}, f_i \in \mathcal{F} \right\}$$

is also.

Proof. Suppsose $f_i \in \mathcal{F}$ (and $f_i' \in \mathcal{F}$) for $i = 1 \ldots n$. Then

$$\sum_{i=1}^{n} c_i \cdot f_i \in \operatorname{span} \mathcal{F}$$

$$\Downarrow$$

$$\left(\sum_{i=1}^{n} c_i \cdot f_i \right)' = \sum_{i=1}^{n} c_i \cdot \underbrace{f_i'}_{\in \mathcal{F}} \in \operatorname{span} \mathcal{F}$$

□

In plain words a d-closed space always can be considered as a linear space (closed under real, linear combination).

A d-closed space $\mathcal{F}$, can be finite, or infinite dimensional. for expamle

$$\mathcal{F} = \operatorname{span} \left\{ e^{\lambda x} | \lambda \in \mathbb{R} \right\}$$

is infinite dimensional, however (somehow) diagonal. And

$$\mathcal{F} = \operatorname{span}\left\{e^{\lambda_i x} | \lambda_i \in \mathbb{R}, i = 1 \ldots n\right\}$$

is n dimensional.

Claim 2.3. *Derivation on an n dimansional, d-closed space ($\mathcal{F}$) can be represented by a $D_{\mathcal{F}} \in \mathbb{R}^{n \times n}$ matrix. We call the matrix D the differential operator in the space $\mathcal{F}$.*

Proof. We know that $\frac{\mathrm{d}}{\mathrm{d}x} : \mathcal{F} \mapsto \mathcal{F}$, since $\mathcal{F}$ is d-closed. We also know that $\mathcal{F}$ is a finite dimensional linear space and derivation is linear, therefore it has to be expressed with a matrix-multiplication.

Constructively, let us choose a basis: $\mathcal{F} = \operatorname{span}\{f_i\}_{i=1\ldots n}$. The j^{th} element in the i^{th} coloumn of the matrix D is the coefficient c_j where

$$f_i' = \sum_{l=1}^{n} c_l \cdot f_l \in \mathcal{F}$$

□

For example let $\mathcal{F}_{\text{trig}} := \{A \sin(x) + B \cos(x) | A, B \in \mathbb{R}\}$. Then $(A \sin + B \cos)' = -B \sin + A \cos$, hence

$$D_{\text{trig}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Claim 2.4. *There are three basic, finite dimensional d-closed spaces:*

- $\mathcal{F}_{\textit{trig},\omega} = \operatorname{span}\{\sin(\omega x), \cos(\omega x)\}$, $\omega \in \mathbb{R}$
- $\mathcal{F}_{\textit{exp},\lambda} = \left\{c \cdot e^{x\lambda} | c \in \mathbb{R}\right\}$, $\lambda \in \mathbb{R}$
- $\mathcal{F}_{\textit{poli},n} = \operatorname{span}\left(\{x^i\}_{i=0\ldots n}\right)$, $n \in \mathbb{N}$

and the derivation acts on these as

- $D_{\textit{trig},\omega} =$

$$\begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \in \mathbb{R}^{2\times 2}$$

- $D_{exp,\lambda} =$
$$\begin{bmatrix} \lambda \end{bmatrix} \in \mathbb{R}^{1\times 1}$$
- $D_{poli,n} =$
$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & & 0 \\ \vdots & & \ddots & \ddots & \\ & & & 0 & n \\ & & & & 0 \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}$$

respectively

Proof. One can check these matrices by direct differentiation. □

There are other d-closed function-spaces. We present two ways of constructing d-closed spaces.

Claim 2.5. *Let* $\mathcal{F} = \operatorname{span}\{f_i\}_{i=1\dots n}$ *and* $\mathcal{G} = \operatorname{span}\{g_j\}_{j=1\dots m}$ *two finite, d-closed space with differential operators* $D_\mathcal{F} \in \mathbb{R}^{n\times n}, D_\mathcal{G} \in \mathbb{R}^{m\times m}$ *respectively. Also suppose that* $\mathcal{F} \cap \mathcal{G} = \emptyset$

Then the space $\mathcal{F} \oplus \mathcal{G} = \operatorname{span}\{\mathcal{F}, \mathcal{G}\} = \{f + g | f \in \mathcal{F}, g \in \mathcal{G}\}$ *is also a finite, d-closed space, with a differential operator:*

$$D = \operatorname{diag}\{D_\mathcal{F}, D_\mathcal{G}\} \in \mathbb{R}^{(n+m)\times(n+m)} \tag{7}$$

Proof. Let $f \in \mathcal{F}, g \in \mathcal{G}$.

$$(f+g)' = f' + g' = \underbrace{D_\mathcal{F} f}_{\in\mathcal{F}} + \underbrace{D_\mathcal{G} g}_{\in\mathcal{G}} \in \mathcal{F} \oplus \mathcal{G} \tag{8}$$

If we consider $\mathcal{F} \oplus \mathcal{G}$ as $\operatorname{span}\{f_1, \dots f_n, g_1, \dots g_m\}$, then the the formula (7) also follows. □

Claim 2.6. *Let* $\mathcal{F} = \operatorname{span}\{f_i\}_{i=1\dots n}$ *and* $\mathcal{G} = \operatorname{span}\{g_j\}_{j=1\dots m}$ *two finite, d-closed space with differential operators* $D_\mathcal{F} \in \mathbb{R}^{n\times n}, D_\mathcal{G} \in \mathbb{R}^{m\times m}$ *respectively.*

Then the space $\mathcal{F} \otimes \mathcal{G} = \operatorname{span}\{f_i \cdot g_j\}_{\substack{i=1\dots n \\ j=1\dots m}}$ *is also a finite, d-closed space, with a differential operator:*

$$D = D_\mathcal{F} \otimes \operatorname{Id}_m + \operatorname{Id}_n \otimes D_\mathcal{G} \quad \in \mathbb{R}^{(n\cdot m)\times(n\cdot m)} \tag{9}$$

Proof. First of all, by $\operatorname{span}\{f_i \cdot g_j\}_{\substack{i=1\ldots n\\ j=1\ldots m}}$ we mean

$$\operatorname{span}\{\underbrace{f_1 \cdot g_1, f_1 \cdot g_2, \ldots f_1 \cdot g_m}_{f_1 \cdot \mathcal{G}}, \underbrace{f_2 \cdot g_1, \ldots, f_2 \cdot g_m}_{f_2 \cdot \mathcal{G}}, \ldots f_n \cdot g_m\}.$$

The ordering matters.

Now

$$(f_i \cdot g_j)' = f_i' \cdot g_j + f_i \cdot g_j' = \underbrace{D_{\mathcal{F}} f_i}_{\in \mathcal{F}} \cdot g_j + f_i \cdot \underbrace{D_{\mathcal{G}} g_j}_{\in \mathcal{G}} \in \mathcal{F} \otimes \mathcal{G} \tag{10}$$

The above formula also consludes the formula (9). $\square$

Example 2.1. *Let $\mathcal{F} := \operatorname{span}\{1, x\} = \mathbb{R}_1[x]$, we have called that $\mathcal{F}_{poli,1}$ earlier. Furthermore, let $\mathcal{G} := \mathcal{F}_{exp,1}$. Then $\mathcal{F} \otimes \mathcal{G} = \operatorname{span}\{e^x, x \cdot e^x\}$ a two dimensional space.*

Let us calculate the differential operator according to (9).

$$D = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \end{bmatrix} = \tag{11}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \tag{12}$$

Let us derivate the function $x \cdot e^x = 0 \cdot e^x + 1 \cdot x \cdot e^x$. This function is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ with vector notation.

$$(x \cdot e^x)' = \begin{bmatrix} e^x & x \cdot e^x \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} e^x & x \cdot e^x \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = (1 + x)e^x$$

Note that the formula (9) of the differential operator can be extended for arbitrary many tensor terms, for three terms i.e.

$$D_{\mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \mathcal{F}_3} = D_1 \oplus D_2 \oplus D_3 = D_1 \otimes \mathrm{Id}_2 \otimes \mathrm{Id}_3 + \mathrm{Id}_1 \otimes D_2 \otimes \mathrm{Id}_3 + \mathrm{Id}_1 \otimes \mathrm{Id}_2 \otimes D_3 \tag{13}$$

and similarly for more terms.

Consider equation (1), where the right-hand-side comes from a closed, finite dimensional space $\mathcal{F}$, with differential operator $D \in \mathbb{R}^{d \times d}$. Then $f(x)$

can be represented by a vector $f(x) \leftrightarrow \underline{v} \in \mathbb{R}^d$ and $\underline{x} \in \mathbb{R}^d$ is sought such that

$$\left(D^n + a_{n-1}D^{n-1} + \dots a_1 D + a_0 \operatorname{Id}\right) \underline{x} = \underline{v} \tag{14}$$

The problem also includes to *find* a finite, d-closed space $\mathcal{F}$, such that $f \in \mathcal{F}$ and (14) can be solved.

Consider Example 1.1. $y'' + y = x$ can be written as

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \operatorname{Id}\right) y = x \tag{15}$$

We have to find a d-closed space $\mathcal{F}$ with differential operator D such that $x \in \mathcal{F}$ and we have to find every solution of the equation

$$\left(D^2 + \operatorname{Id}\right) \underline{y} = \underline{x} \tag{16}$$

where $\underline{x}$ is the vector representation of the function x and the vector $\underline{y}$ is the variable.

Recall the solution of this simple case. $y(x) = y_h(x) + y_{ip}(x) = c_1 \sin(x) + c_2 \cos(x) + x$. The space $\mathcal{F}$ is $\operatorname{span}\{1, x, \sin(x), \cos(x)\}$. The differential operator is

$$D = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \tag{17}$$

Mind the direct product structure of $\mathcal{F}$.

$$D^2 + \operatorname{Id} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{18}$$

Therefore every solution of $(D^2 + \operatorname{Id})\underline{y} = (0, 1, 0, 0)^\top$ has the form $(0, 1, c_1, c_2)^\top$. This is exactly what we have already seen.

3 Homogeneous case

Consider

$$y^{(n)}(x) + a_{n-1} \cdot y^{(n-1)}(x) + \ldots + a_1 \cdot y'(x) + a_0 \cdot y(x) = 0 \tag{19}$$

Claim 3.1. *The solutions of (19) form a finite, d-closed space.*

Proof. If $f(x)$ solves (19), then $f \in \mathcal{F}$.

- $\mathcal{F}$ is d-closed.

$$f^{(n)} + a_{n-1} \cdot f^{(n-1)} + \ldots a_1 \cdot f' + a_0 \cdot f = 0 \tag{20}$$

$$\Downarrow$$

$$\left(f^{(n)} + a_{n-1} \cdot f^{(n-1)} + \ldots a_1 \cdot f' + a_0 \cdot f\right)' = 0 \tag{21}$$

$$\Updownarrow$$

$$(f')^{(n)} + a_{n-1} \cdot (f')^{(n-1)} + \ldots a_1 \cdot f'' + a_0 \cdot f' = 0 \tag{22}$$

- Linear combination of solutions is also a solution. This is trivial.

- Finite dimensionality follows from existence and uniqueness theorems. Without initial or bondary conditions, the degree of freedom is at most n in an n^{th} order differential equation.

$\square$

So we search for an n dimensional, d-closed space $\mathcal{F}$, with differential operator D such that

$$D^n + a_{n-1} D^{n-1} + \ldots + a_1 D + a_0 \,\mathrm{Id} = 0. \quad \in \mathbb{R}^{n \times n} \tag{23}$$

In other words, the task is to find a differential operator which has a given spectrum (characteristic polynomial).

Claim 3.2. *Let $D \in \mathbb{R}^{n \times n}$ be any given matrix. Then D is the differential operator of a finite, d-closed space: $\mathcal{F}$, and $\dim \mathcal{F} \leq n$.*

Proof. Consider the following first order system of differential equations

$$\underline{f}'(x) = D\underline{f}(x) \tag{24}$$

where $\underline{f}(x) = \left(f_1(x), \ldots f_n(x)\right)^\top$.

Similarly like in Claim 3.1, we can state that the coordinate functions of the solutions form a (maximum) n dimensional d-closed space

$$\mathcal{F} := \operatorname{span}\left\{f_i(x), i = 1 \ldots n \text{ such that } \left(f_1(x), \ldots f_n(x)\right)^\top \text{ solves } (24)\right\}. \tag{25}$$

The differential operator of this space is D, this is exactly what (24) means. The functions in $\mathcal{F}$ are uniquely detrermined up to linear combination.

We can also state that the Jordan normal form of D is the only significant factor, because change of basis can be represented as $\underline{f} \rightsquigarrow P\underline{f}$, $D \rightsquigarrow PDP^{-1}$ and $\operatorname{span}\{\underline{f}\} = \operatorname{span}\{P\underline{f}\}$. □

We saw that D is not only a property of a finite, d-closed space, rather identifies it.

Consequence 3.3. *The only essential, finite, d-closed spaces are in Claim 2.4.*

Proof. Technically, we have to check, that one can construct any matrix (any given Jordan normal form) with the matrices $D_{\text{poli},n}$, $D_{\exp,\lambda}$ and $D_{\text{trig},\omega}$. If so, then there are no more significantly different d-closed spaces.

- Consider the differential operator of $\mathcal{F}_{\exp} \otimes \mathcal{F}_{\text{trig}}$:

$$D_{\exp,\lambda} \otimes \mathrm{Id}_2 + \mathrm{Id}_1 \otimes D_{\text{trig},\omega} =$$
$$\begin{bmatrix}\lambda\end{bmatrix} \otimes \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \begin{bmatrix}1\end{bmatrix} \otimes \begin{bmatrix}0 & \omega\\ -\omega & 0\end{bmatrix} = \begin{bmatrix}\lambda & \omega\\ -\omega & \lambda\end{bmatrix} \tag{26}$$

  The eigenvalues of this matrix are $\lambda \pm i\omega$. This means that any complex number can be the eigenvalue of a differential operator (we just gave the space and the operator).

- One can concatenate the blocks above. In this way, any diagonal matrix can get as a differential operator. What about the Jordan blocks?

- Suppose that we have a d-closed space $\mathcal{F}$ with differential operator $D_{\mathcal{F}}$. Let us calculate the differential operator of $\mathbb{R}_n[x] \otimes \mathcal{F}$:

$$\underbrace{D_{\text{poli},n} \otimes \mathrm{Id}}_{\begin{bmatrix} 0 & \mathrm{Id} & & \\ & \ddots & \ddots & \\ & & 0 & n\,\mathrm{Id} \\ & & & 0 \end{bmatrix}} \quad + \quad \underbrace{\mathrm{Id} \otimes D_{\mathcal{F}}}_{\begin{bmatrix} D_{\mathcal{F}} & & \\ & \ddots & \\ & & D_{\mathcal{F}} \end{bmatrix}} \quad = \tag{27}$$

$$\begin{bmatrix} D_{\mathcal{F}} & \mathrm{Id} & 0 & \cdots & 0 \\ 0 & D_{\mathcal{F}} & 2\,\mathrm{Id} & & 0 \\ \vdots & & \ddots & \ddots & \\ & & & D_{\mathcal{F}} & n\,\mathrm{Id} \\ & & & & D_{\mathcal{F}} \end{bmatrix} \tag{28}$$

- For $D_{\mathcal{F}} = [\lambda] \in \mathbb{R}^{1\times 1}$ this is exactly a Jordan block. For $\mathcal{F} = \mathcal{F}_{\exp,\lambda} \otimes$

$\mathcal{F}_{\mathrm{trig},\omega}$ it is

$$\begin{bmatrix} \lambda & \omega & 1 & 0 & & \\ -\omega & \lambda & 0 & 1 & & \\ & & \lambda & \omega & 2 & 0 \\ & & -\omega & \lambda & 0 & 2 \\ & & & & \ddots & & \ddots \end{bmatrix} \underset{\text{conjugation transform } P\bullet P^{-1}}{\rightsquigarrow} \tag{29}$$

$$\begin{bmatrix} \lambda+i\omega & 0 & 1 & 0 & & \\ 0 & \lambda-i\omega & 0 & 1 & & \\ 0 & 0 & \lambda+i\omega & 0 & 1 & 0 \\ 0 & 0 & 0 & \lambda-i\omega & 0 & 1 \\ & & & & \ddots & & \ddots \end{bmatrix} \underset{\text{exchanging the subspaces}}{\rightsquigarrow} \tag{30}$$

$$\begin{bmatrix} \lambda+i\omega & 1 & 0 & 0 & & & & \\ 0 & \ddots & \ddots & 0 & & & & \\ & & \lambda+i\omega & 1 & & & & \\ & & & \lambda+i\omega & & & & \\ & & & & \lambda-i\omega & 1 & 0 & 0 \\ & & & & 0 & \ddots & \ddots & 0 \\ & & & & & & \lambda-i\omega & 1 \\ & & & & & & & \lambda-i\omega \end{bmatrix} \tag{31}$$

$$\underset{\text{In the characteristic polynomial}}{\rightsquigarrow} \qquad p(z) = \left((\lambda-z)^2+\omega^2\right)^{n+1} \tag{32}$$

Finaly we showed that any real polynomial can get as a characteristic polinom of a differential operator.

In more practical context: let

$$p(x) = \prod_{i=1}^{k'} (x-x_i)^{\nu_i} \cdot \prod_{j=k'+1}^{k} \left((x-x_j)^2+c_j\right)^{\nu_j}.$$

given with $c_j > 0$. For the terms $(x-x_i)^{\nu_i}$ take $\mathbb{R}_{\nu_i-1}[x] \otimes \mathcal{F}_{\mathrm{exp},x_i}$. For the irreducible blocks take the $\mathbb{R}_{\nu_j-1}[x] \otimes \mathcal{F}_{\mathrm{exp},x_j} \otimes \mathcal{F}_{\mathrm{trig},\sqrt{c_j}}$. And concatenate the blocks diagonally (take the direct sum of the d-closed spaces). $\square$

Remark 3.4. *The characteristic polynomial of a differential operator is exactly it's minimal polynomial.*

Proof. Consider a Jordan block of size $m+1$ in D with eigenvalue λ:

$$\begin{bmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \ddots & \\ & & & \lambda \end{bmatrix} \tag{33}$$

This gives us the subspace $e^{\lambda x} \cdot \mathbb{R}_m[x]$, therefore the additional smaller Jordan blocks do not extend this subspace, $e^{\lambda x} \cdot \mathbb{R}_l[x] \subseteq e^{\lambda x} \cdot \mathbb{R}_m[x]$ if $l \leq m$. The spanned space, constructed in (25), is effected only by the largest Jordan block. For trigonometrical subspaces likewise.

In this way, every occurance of an eigenvalue can be reduced to the largest Jordan block. Hence the minimal polinomial is the characteristic polynomial. □

Also one can construct a differential operator of the form

$$\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

by taking the generating system (not basis) $\{e^{\lambda x}, x \cdot e^{\lambda x}, e^{\lambda x}\}$. But in this set up, the differential operator is not well defined.

We shortly summerize the results about *finite dimensional*, d-closed spaces:

- Every d-closed space has a differential operator.
- Every matrix can be a differential operator of a d-closed space.
- The differential operator identifies the d-closed space.
- We gave the d-closed space of every matrix.

Now we can solve (19).

Claim 3.5 (solution of the homogeneous case)**.** *We constructed all solutions of the ODE (19) in Consequence 3.3.*

Proof. Find the d-closed space, corresponding to the matrix with characteristic polynomial $x^n + a_{n-1}x^{n-1} + \dots a_1 x + a_0$. This was constructed in Consequence 3.3. There are no more solutions, since the space constructed in Consequence 3.3. was n dimensional, closed and any other space with bigger dimension would contradict the uniqueness of the solution of an ODE. □

4 Inhomogeneous case

Let us call the solution space of the homogeneous equation $\mathcal{F}_{\text{hom}}$ with differential operator D_{hom} (this is what we constructed in the former section). From earlier results we know that $\mathcal{F}_{\text{hom}}$ is well defined and n dimensional.

Now we consider (1) again and let us call the polynomial on the left-hand-side

$$p(x) := x^n + a_{n-1}x^{n-1} + \dots a_1 x + a_0. \tag{34}$$

With the d-closed space technique we are able to solve (1) when the right-hand-side comes from a d-closed space. In finite dimensional case, f can only contain exponential, trigonometrial and/or polinomial terms.

Due to the block structure of differential operators (Jordan blocks of different eigenvalues), we only deal with the case

$$f(x) = x^m \cdot e^{\lambda x} \cdot (A \sin(\omega x) + B \cos(\omega x)) \tag{35}$$

where $m \in \mathbb{N}, \lambda \in \mathbb{R}, \omega \in \mathbb{R}$.

Claim 4.1. *Assume, that the function (35) is not in the space of the homogeneous solutions ($\mathcal{F}_{hom}$). Take $\mathcal{F} = \mathcal{F}_{poli,m} \otimes \mathcal{F}_{exp,\lambda} \otimes \mathcal{F}_{trig,\omega}$ with differential operator $D \in \mathbb{R}^{2(m+1)\times 2(m+1)}$. Then*

$$\det p(D) \neq 0 \tag{36}$$

$$\Downarrow$$

$$p(D)\underline{y} = \underline{f} \tag{37}$$

can be solved uniquelly, where $\underline{f}$ is the vector representation of the function f in the space $\mathcal{F}$. The function represented by the vector $\underline{y}$ solves (1).

Proof. The assumption of the statement is equivalent with the fact that $\lambda \pm i\omega$ in (35) is not the eigenvalue of the differential operator D_{hom}.

If so, then $p(D)$ is clearly invertible, since $p(x)$ is the characteristic polynomial of D_{hom} and D has no joint eigenvalue with D_{hom}.

Therefore, the function $p(D)^{-1}\underline{f}$ solves (1), this is exactly (37). $\square$

Now let us assume, that the function in (35) is in $\mathcal{F}_{\text{hom}}$. We suppose that $\mathcal{F}_{\text{hom}} = \mathcal{F}_{\text{poli},k} \otimes \mathcal{F}_{\text{exp},\lambda} \otimes \mathcal{F}_{\text{trig},\omega}$ without the loss of generality (mind the Jordan blocks). Then $p(x) = \left((x-\lambda)^2 + \omega^2\right)^k$ and $p(D)$ is not invertible. $\mathcal{F}$ and $\mathcal{F}_{\text{hom}}$ differs only in the order of the polynomial term.

Claim 4.2. *Let D_j be the differential operator of the space $\mathcal{F}_{poli,j} \otimes \mathcal{F}_{exp,\lambda} \otimes \mathcal{F}_{trig,\omega}$, the characteristic polynomial of D_j is $p_j(x) = \left((x-\lambda)^2 + \omega^2\right)^{j+1}$. Then for the image the following holds:*

$$\operatorname{Im}\left(p_j(D_{j+m})\right) = \mathcal{F}_{poli,m} \otimes \mathcal{F}_{exp,\lambda} \otimes \mathcal{F}_{trig,\omega} \tag{38}$$

for any $j \in \mathbb{N}$.

Proof. Recall the form of D_j in (29), there ar $j+1$ blocks in the diagonal. We can calculate $p_j(D_{j+m})$ directly.

$$\left(\left(\begin{bmatrix} \lambda & \omega & 1 & 0 & & & \\ -\omega & \lambda & 0 & 1 & & & \\ & & \lambda & \omega & 2 & 0 & \\ & & -\omega & \lambda & 0 & 2 & \\ & & & & \ddots & & \ddots \end{bmatrix} - \lambda \operatorname{Id}\right)^2 + \omega^2 \operatorname{Id}\right)^{j+1} = \tag{39}$$

$$\left(\begin{bmatrix} 0 & \omega & 1 & 0 & & & \\ -\omega & 0 & 0 & 1 & & & \\ & & 0 & \omega & 2 & 0 & \\ & & -\omega & 0 & 0 & 2 & \\ & & & & \ddots & & \ddots \end{bmatrix}^2 + \omega^2 \operatorname{Id}\right)^{j+1} = \dots \tag{40}$$

We use the 2×2 block structure.

$$\dots = \left(\begin{bmatrix} D_{\mathrm{trig},\omega} & \mathrm{Id} & & \\ & D_{\mathrm{trig},\omega} & 2\,\mathrm{Id} & \\ & & \ddots & \ddots \end{bmatrix}^2 + \omega^2\,\mathrm{Id} \right)^{j+1} = \tag{41}$$

$$\left(\begin{bmatrix} D^2_{\mathrm{trig},\omega} & 2D_{\mathrm{trig},\omega} & 2\,\mathrm{Id} & & \\ & D^2_{\mathrm{trig},\omega} & 4D_{\mathrm{trig},\omega} & 6\,\mathrm{Id} & \\ & & D^2_{\mathrm{trig},\omega} & 6D_{\mathrm{trig},\omega} & 12\,\mathrm{Id} \\ & & \ddots & \ddots & \ddots \end{bmatrix} + \omega^2\,\mathrm{Id} \right)^{j+1} = \dots \tag{42}$$

Like the square of a Jordan block. And we know that $D^2_{\mathrm{trig},\omega} = \begin{bmatrix} -\omega^2 & 0 \\ 0 & -\omega^2 \end{bmatrix}$.

$$\dots = \begin{bmatrix} 0 & 2D_{\mathrm{trig},\omega} & 2\,\mathrm{Id} & & \\ & 0 & 4D_{\mathrm{trig},\omega} & 6\,\mathrm{Id} & \\ & & 0 & 6D_{\mathrm{trig},\omega} & 12\,\mathrm{Id} \\ & & \ddots & \ddots & \ddots \end{bmatrix}^{j+1} \tag{43}$$

The matrix under the power is nilpotent, and every power decreases the rank by 2 (1 block). Therefore the $j+m$ blocks in the upper-diagonal collapses into m blocks in the $j+1^{\mathrm{th}}$-diagonal. Hence the image space is the upper $2(m+1)$ component, which is the subspace $\mathbb{R}_m[x] \otimes \mathcal{F}_{\exp,\lambda} \otimes \mathcal{F}_{\mathrm{trig},\omega}$. $\square$

On one word, to get (35) on the right hand side, multiply the test-function-space by $\mathbb{R}_{k-1}[x]$, where k is the multiplicity of the coincide root in the homogeneous solution.

Since the range space in Claim 4.1 and 4.2 is the same: $\mathbb{R}_m[x] \otimes \mathcal{F}_{\exp,\lambda} \otimes \mathcal{F}_{\mathrm{trig},\omega}$, and due to the collapse of the Jordan blocks in Claim 4.2, the inhomogeneous particular solution is also unique.

5 In infinite dimension

With the help of characterising the finite dimensional d-closed spaces, we found the unique solution of the inhomogeneous equation, if the right-hand-side comes from a finite space. But what if it comes form an infinite dimensional d-closed space? What can we say about infinite dimensional d-closed spaces?

Definition 2.1 allows infinite dimensional spaces and Claim 2.2 also holds. Moreover the differential operator also exists, but it is not a matrix. In this way equation (37) makes sense, but $\underline{y}$ and $\underline{f}$ are infinite dimensional vectors and D is an operator.

Technically $C^\infty(\mathbb{R})$ is a d-closed space, therefore no matter what is the right-hand-side, the infinite dimensional formalism always holds. Constructively, let $f(x) \in C^\infty(\mathbb{R})$ and take $\mathcal{F}^n := \operatorname{span}\{f, f', \dots f^{(n)}\}$. If $\mathcal{F}^m \subseteq \mathcal{F}^n$ for some $m > n$, then we are in the finite dimensional case. The general case can be solved with the Wronskian, and there is no need for further discussion. However, we show an example of our linear algebraic method in infinite dimensional case.

Example 5.1. *Let us consider*

$$y''(x) + y(x) = e^{(x^2)} \tag{44}$$

In more general consider the d-closed space $\mathbb{R}[x] \cdot e^{(x^2)}$*. For a* $q(x) \in \mathbb{R}[x]$*:*

$$\left(q(x) \cdot e^{(x^2)}\right)' = \left(q'(x) + 2x \cdot q(x)\right) e^{(x^2)} \tag{45}$$

If one consider $\mathbb{R}[x] \cdot e^{x^2}$ *as:*

$$\operatorname{span}\left\{e^{x^2}, x \cdot e^{x^2}, x^2 \cdot e^{x^2}, \dots\right\} \tag{46}$$

then one can check that the differential operator is:

$$\mathcal{D} = \begin{bmatrix} 0 & 1 & 0 & \cdots & & \\ 2 & 0 & 2 & & & \\ 0 & 2 & 0 & 3 & & \\ 0 & 0 & 2 & 0 & 4 & \\ & & & \ddots & 0 & \ddots \end{bmatrix} \tag{47}$$

Let us cut the differential operator.

$$D_n := \begin{bmatrix} 0 & 1 & 0 & \cdots & & \\ 2 & 0 & 2 & & & \\ 0 & 2 & 0 & 3 & & \\ 0 & 0 & 2 & 0 & 4 & \\ & & & \ddots & 0 & \ddots \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)} \tag{48}$$

Then $(D_n^2 + \mathrm{Id})\,\underline{y} = (1, \underbrace{0, \dots 0}_{n})^\top$ *can be solved for* $\underline{y} \in \mathbb{R}^{n+1}$*. If the operator* $D_n^2 + \mathrm{Id}$ *has a uniform spectral gap, then its inverse is uniformly bounded for every* n*, thus*

$$\left(D_n^2 + \mathrm{Id}\right)^{-1} (1, 0, \dots 0)^\top \tag{49}$$

converges to an ℓ_2 *vector, the solution of (44).*