THESIS

Application of an Abstract Multiplier Method in Model Predictive Control with Guaranteed Cost

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Contents

Abstract

In this work we consider a certain class of continuous-time, uncertain systems having the following properties

- quadratically constrained nonlinearity/uncertainty,
- non-accessible state space,
- external disturbances and
- quadratic cost function.

These specifications cover several frequently investigated types of uncertain systems. We propose a nonlinear model predictive control (NMPC) and prove that the applied control policy gives an upper bound of the cost function i.e. a cost guaranteeing control is constructed. For a certain class of disturbances we show that the proposed controller gives a closed-loop system with locally asymptotically stable equilibrium.

Due to the non-accessible state space, we use dynamic output feedback control what we keep quadratically constrained. The abstract multiplier method is used to obtain matrix inequalities that give necessary and sufficient condition for the existence and the computation of a quadratic Lyapunov function. The matrix inequalities are converted into linear matrix inequalities (LMI) in order to obtain a computationally much more effective way for the control design though this latter inequalities provide only a sufficient condition for the existence of the Lyapunov function. The application of the model predictive control makes it possible to improve the quality of the controlled system. Namely, the speed of the convergence to the equilibrium can be increased in comparison with the initially computed dynamic output feedback. Simultaneously, the overall cost of the process is reduced, as well.

A novel element of the proposed method is the application of the NMPC technique with non-accessible state space.

Kivonat

A dolgozat a folytonos idejű rendszerek egy bizonyos osztályával foglalkozik szerepeltetve

- kvadratikusan korlátozott bizonytalanságokat,
- az állapot helyett egy output mérhetőségét,
- külső perturbációs függvényeket és
- kvadratikus célfüggvényt.

Ezek a tulajdonságok az irodalomban tárgyalt bizonytalan rendszerek széles körét lefedik. Bemutatunk egy NMPC szabályzót és felső korlátot bizonyítunk a célfüggvényre, vagyis garantált költségű szabályzót konstruálunk. Megmutatjuk, hogy a perturbációk egy osztálya esetén ez a szabályozó biztosítja az egyensúlyi helyzet lokális aszimptotikus stabilitását is.

Feltételezük, hogy az állpot helyett csak egy output hozzáférhető a visszacsatoláshoz, ezért dinamikus output-visszacsatolást alkalmazunk, mely kvadratikus korlátnak engedelmeskedik. Az absztrakt multiplikátor módszer használatával mátrix egyenlőtlenségeket vezetünk le, amelyek kvadratikus Lyapunov függvény létezésének szükséges és elégséges feltételeit adják. Ezen egyenlőtlenségeket átalakítjuk lineáris mátrix egyenlőtlenségekké (LMI). Bár ezek az egyenlőtlenségek a Lyapunov függvény létezésének csak elégséges feltételeit adják, de megoldásuk számításigénye kisebb. Az NMPC technikának köszönhetően képesek vagyunk futás közben javítani a vezérelt rendszer minőségét, nevezetesen gyorsabb konvergenciát tudunk elérni, mint ha csak a kezdő időpillanatban kiszámított dinamikus visszacsatolást használnánk, egyben csökkentjük a költség felső korlátját is.

A dolgozatban újdonságként szerepel az NMPC szabályzó nem állapot-visszacsatolással, hanem dinamikus output-visszacsatolással történő alkalmazása.

Nomenclature

$\mathbb{R}_+$ is $[0, \infty)$, usually time

$A^\top$ is the transpose of the real matrix A

$A > 0$ where $A = A^\top \in \mathbb{R}^{n\times n}$, means that A is positive definite

$AB = A \cdot B$ every product is a matrix product of appropriate sized matrices

$\langle x; y\rangle = x^\top \cdot y$ scalar product of vectors with equal dimensions

$(*)AB$ $= B^\top AB$, where $B \in \mathbb{R}^{n\times m}, A = A^\top \in \mathbb{R}^{n\times n}$. In other words $(*)$ completes the symmetric expression.

$\|x\|_A$ $= \sqrt{x^\top Ax}$ where $x \in \mathbb{R}^n$ and $A > 0$

$\Im z/\Re z$ is the imaginary/real part of $z \in \mathbb{C}$

$\operatorname{Im} A$ is the range of $A \in \mathbb{R}^{m\times n}$ i.e. $\{Ax \in \mathbb{R}^m |\, x \in \mathbb{R}^n\}$

$\operatorname{Ker} A$ is the kernel of $A \in \mathbb{R}^{m\times n}$ i.e. $\{x \in \mathbb{R}^n |\, Ax = 0 \in \mathbb{R}^m\}$

$A^\perp$ where $A \in \mathbb{R}^{m\times n}$, is an orthonormal basis of $\operatorname{Ker} A$ i.e. $A^\perp \in \mathbb{R}^{n\times k}$ where $k = \dim \operatorname{Ker} A$ and $A \cdot A^\perp = 0 \in \mathbb{R}^{m\times k}$.

$\operatorname{rank} A$ $= \dim \operatorname{Im} A \le \min\{m, n\}$, the rank of $A \in \mathbb{R}^{m\times n}$

row rank is the number of independent rows of $A \in \mathbb{R}^{m\times n}$. It is complete, if it is equal to m.

column rank is the number of independent columns of $A \in \mathbb{R}^{m\times n}$. It is complete, if it is equal to n.

I_l is the identity matrix of dimension l. If the dimension is unequivocal then the index l is omitted.

$\operatorname{span}(S)$ is the linear closure of a set S.

$X \rightsquigarrow A^\top XA$ is a *congruence transformation* on the symmetric matrix X, with the invertible matrix A. Note that matrix positivity is invariant under this transformation.

$\sigma(A)$ $\subseteq \mathbb{C}$ is the spectrum of the square matrix $A \in \mathbb{R}^{n\times n}$.

Chapter 1

Preliminaries

This work is devoted to a certain problem of control theory. In control theory the object of interest is a *system* to be controlled. The system has well defined dynamics (of its own) and we have some pre-defined ways to interfere (*control*). As an optimization problem has an objective function, a control problem has a *control task*. This can be a desired state of the system to reach, an objective function (functional) to extremize or a given bound to regard.

In this work the system is usually described by a differential equation and the control is the ability to change some parameters on the right-hand side.

$$\dot{x}(t) = f(t, x(t), u(t))$$

where $\mathcal{X} \subseteq \mathbb{R}^n$ is the state space, $x \in \mathcal{X}$ is the state of the system and $t \in \mathbb{R}_+$ is the time. The control u comes from a given set $u \in \mathcal{U} \subseteq \mathbb{R}^m$. Once a function $u(\cdot) : \mathbb{R}_+ \mapsto \mathbb{R}^m$ is given, the system behaves according to $\dot{x}(t) = F(t, x(t)) = f(t, x(t), u(t))$. The solution operator is denoted by $\xi(t; t_0, x_0, u(\cdot))$ i.e. the integral curve starting from $x(t_0) = x_0$ controlled by the function $u(\cdot)$. Our task is to choose or compute u with which the system behaves in the desired way.

The control function is restricted to a certain set of functions: $\Delta \subseteq \mathcal{U}^{\mathbb{R}_+}$, $u(\cdot) \in \Delta$. The control functions in Δ are called *admissible control*. Usually Δ is the set of smooth or measurable functions.

From now on a control problem is defined with the above emphasized four components.

1.1 Introduction

1.1.1 Basic examples, stability

In this section one can see some well known control problems, see [7]. These examples are here for the better understanding of the main phenomena.

One of the most basic examples in control theory is the stabilization of a linear system. Let the system be described by

$$\dot{x} = Ax + Bu, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are given matrices, $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control. The control task is to find a matrix $K \in \mathbb{R}^{m\times n}$ such that the linear *state-feedback* control $u = Kx$ asymptotically stabilizes the constant 0 solution. The *closed loop system* can be obtained by substituting u.

$$\dot{x} = Ax + B\underbrace{Kx}_{u}$$

One can see that the task is to find K such that

$$\sigma(A + BK) \subseteq \{z \in \mathbb{C} | \Re z < 0\}. \tag{1.2}$$

There is an other problem, namely *pole placement*, when one wishes to control the whole spectrum of $A + BK$.
Consider matrices $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$ and a given set of n complex numbers (paired by conjugation) $\Lambda = \{\lambda_i : i = 1 \dots n\} = \{\overline{\lambda_i} : i = 1 \dots n\} \subseteq \mathbb{C}$. The problem is to find a $K \in \mathbb{R}^{m\times n}$ such that

$$\det(A + BK - \lambda I) = \prod_{i=1}^{n}(\lambda_i - \lambda). \tag{1.3}$$

If the pole placement can be solved for any spectrum, then (1.2) can be solved also.

Definition 1.1.1. *Consider the linear system (1.1) with given matrices A, B. The system is called* completely controllable *on the time interval $[t_0, t_1]$, if for any $x_0, x_1 \in \mathbb{R}^n$ a control function $u : [t_0, t_1] \mapsto \mathbb{R}^m$ can be given such that the solution of (1.1) satisfies $\xi(t_1; t_0, x_0, u(\cdot)) = x_1$.*

In other words, one can steer the system from any state into any other state with an admissible control.

Theorem 1.1.2. *A system $\dot{x} = Ax + Bu$ with given matrices $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$ is completely controllable on any time interval $[t_0, t_1]$ with positive length, iff*

$$\operatorname{rank}\left[B, AB, \dots, AB^{n-1}\right] = n. \tag{1.4}$$

Moreover, the pole placement can be solved, iff (1.4) holds true.

Let us define the system

$$\dot{x} = Ax + Bu \tag{1.5}$$
$$y = Cx$$

where x, u are the same as above and $y \in \mathbb{R}^p$ is the available output. In this case one can only get information about x via y, typically $p < n$.

Definition 1.1.3. *The system (1.5) with given matrices A, B, C is* completely observable *on the time interval $[t_0, t_1]$, if the unknown initial condition $x(t_0) = x_0$ is determined by the output and control functions: $y : [t_0, t_1] \mapsto \mathbb{R}^p, u : [t_0, t_1] \mapsto \mathbb{R}^m$.*

Theorem 1.1.4. *The linear system (1.5) with given matrices A, B, C is completely observable on any time interval $[t_0, t_1]$ of positive length, iff*

$$\operatorname{rank} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$$

no matter what B is.

An *observer* can be built to estimate the original state of a completely observable system.

Definition 1.1.5. *Consider the system (1.5) with given matrices A, B, C. Let us define the system*

$$\dot{z} = Fz + Gu + Hy \tag{1.6}$$

where $z \in \mathbb{R}^n$. The system (1.6) is called an observer *of (1.5) if for any given initial condition $x(0) = x_0, z(0) = z_0$ and any admissible control $u(\cdot)$ the corresponding solutions $x(\cdot)$ and $z(\cdot)$satisfy*

$$\lim_{t\to\infty} x(t) - z(t) = 0.$$

This means that the observer can asymptotically reconstruct the state of an observable system using only the output y. One way to build an observer of a completely observable system is described in the following theorem.

Theorem 1.1.6 (Luenberger observer)**.** *Let us define the linear systems*

$$\dot{x} = Ax + Bu, \;\; y = Cx \tag{1.7}$$

with $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}$ satisfying Theorem 1.1.4 and

$$\dot{z} = (A - HC)z + Bu + Hy. \tag{1.8}$$

Then exists a matrix $H \in \mathbb{R}^{p\times n}$ such that (1.8) is an observer of (1.7). Moreover, the choice of H is independent from B.

Let us consider the stabilization of completely observable systems, where the state is not directly accessible.

Theorem 1.1.7 (Separation principle)**.** *Consider the completely controllable and observable system $\dot{x} = Ax + Bu$, $y = Cx$ with given matrices $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}$ (satisfying the conditions of Theorem 1.1.2 and 1.1.4). Obtain The Luenberger observer $\dot{z} = (A - HC)z + Bu + Hy$ as given in Theorem 1.1.6 and the stabilizing, linear state feedback control $u = Kx$, with K satisfying (1.2).*
Then the closed loop system

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} A & BK \\ HC & A - HC + BK \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix}$$

is stable.

Notice, that the control $u = Kz$ is used instead of the state feedback control $u = Kx$. The separately built observer and control together stabilizes the system. The Separation principle is valid only for linear systems. In general one has to treat the observer and the control simultaneously.

1.1.2 Optimal control and Hamilton–Jacobi–Bellman equation

In the former section we focused on finding a stabilizing control. Now an optimal control is sought. In order to talk about optimality, an objective is needed.

Consider the system

$$\dot{x}(t) = f(t, x(t), u(t)) \tag{1.9}$$

where $x \in \mathcal{X} \subseteq \mathbb{R}^n$ is the state space and the time t is in a given time interval $\mathcal{I} = [t_0, t_1]$. The control constraint is $\Delta \subseteq \mathcal{U}^{\mathcal{I}}$ where $\mathcal{U} \subseteq \mathbb{R}^m$ is a given set. The dynamics is defined by the function $f : \mathcal{I} \times \mathcal{X} \times \mathcal{U} \mapsto \mathbb{R}^n$.

The solution operator is ξ i.e. $\xi(t; t_0, x_0, u(\cdot))$ is the solution of (1.9) with initial condition $x(t_0) = x_0$, with a given control $u(\cdot) \in \Delta$, at time $t \in \mathcal{I}$.

The cost functional is given as

$$J(t_0, t_1, x, u(\cdot)) := \underbrace{G\left(\xi(t_1; t_0, x, u(\cdot))\right)}_{\text{terminal penalty}} + \int_{t_0}^{t_1} \underbrace{f_0\left(t, \xi(t; t_0, x, u(\cdot)), u(t)\right)}_{\text{running cost}} \mathrm{d}t \tag{1.10}$$

The control task is to find a control $u \in \Delta$ which minimizes J.

$$\left.\begin{array}{c}\arg\min_{u\in\Delta} \\ \min_{u\in\Delta}\end{array}\right\} J(t_0, t_1, x_0, u(\cdot)) = ?$$

Let us define the *value function* $V : \mathcal{I} \times \mathcal{X} \mapsto \mathbb{R}$ as

$$V(t, x) := \inf_{u\in\Delta} \; J(t, t_1, x, u(\cdot)) \tag{1.11}$$

With this definition $V(t, x)$ gives the optimal cost on $[t, t_1)$, starting from $x(t) = x$, if an optimal control exists.

Within this setup, the Hamilton–Jacobi–Bellman equation (a PDE for V) is in the center of interest.

$$\left.\begin{array}{l}\frac{\partial V}{\partial t}(t, x) = -\min_{u\in\mathcal{U}} \; \{f_0(t, x, u) + \nabla_x V(t, x) \cdot f(t, x, u)\} \\ V(t_1, x) = G(x)\end{array}\right\} \tag{1.12}$$

With some regularity and convexity conditions (see [7]) the HJB equation is an equivalent condition of the optimality.

Note that this formalism does not allow optimization over infinite time intervals, such as $[0, \infty)$. The HJB equation can guarantee optimal control on a finite interval, but we have no influence on the system after time t_1. Therefore stabilizing is an interesting question within this setup.

1.1.3 Lyapunov function

Theorem 1.1.8 (Lyapunov)**.** *Consider the system*

$$\dot{x} = f(x) \in \mathbb{R}^n.$$

Suppose that $f(0) = 0$*. If there exists a continuously differentiable function* $V : \mathbb{R}^n \mapsto \mathbb{R}_+$ *and a neighbourhood* $0 \in \mathcal{N} \subseteq \mathbb{R}^n$ *such that within* $\mathcal{N}$

- $V(x) = 0 \Leftrightarrow x = 0$
- $x \neq 0 \rightarrow \langle \nabla V(x); f(x) \rangle < 0$

then the equilibrium is asymptotically stable.

The neighbourhood $\mathcal{N}$ is called the *basin of attraction*. Our task is to find an appropriate function V in order to prove stability.

1.1.4 Uncertain systems

In some applications there are some unknown parameters on the right-hand side of the differential equation, these are the uncertainties. There are two major types of uncertainties: deterministic or stochastic. Stochastic uncertainty can be caused by improper or noisy measurement data or any stochastic impacts. A deterministic uncertainty can be nonlinearity or some neglected effects.

Dealing with uncertain systems some information about the uncertainty is always necessary. This can be some norm bound, or bounded derivative, or some relation between the unknown term and the state of the system.

Parameter uncertainty

Consider a simple uncertain system:

$$\dot{x} = (A + \Delta A)x + Bu \tag{1.13}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control ($A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are given) and $\Delta A \in \mathbb{R}^{n\times n}$ is the *parameter uncertainty*. The uncertainty ΔA may depend on x, u, on time or it can be a random noise. This means that the elements of the coefficient matrix are known up to some error, suppose that $\|\Delta A\| < \delta$ with a given matrix norm, and a given $0 < \delta$.

One possible way of stabilizing the above system is to find linear state feedback control $u = Kx$ with $K \in \mathbb{R}^{m\times n}$ and a positive definite, quadratic Lyapunov function $V(x) = x^\top Px$ for the closed loop system. The closed loop system is

$$\dot{x}(t) = (A + \Delta A(t, x) + BK)\, x(t) =: f(t, x) \tag{1.14}$$

and the system derivative of the Lyapunov function is

$$\dot{V}(x(t)) = \nabla V(x(t)) \cdot \dot{x}(t) = 2x(t)^\top Pf(t, x) =$$
$$x^\top \begin{pmatrix} I \\ f(t,x) \end{pmatrix}^\top \begin{pmatrix} 0 & P \\ P & 0 \end{pmatrix} \begin{pmatrix} I \\ f(t,x) \end{pmatrix} x.$$

A sufficient condition can be derived for the stabilizing control by substituting (1.14) into $f(t, x)$.

$$\exists\ P > 0, Q > 0 \text{ and } K \text{ s.t. } \forall \Delta A$$
$$(A + BK + \Delta A)^\top P + P(A + BK + \Delta A) \leq -Q$$

Nonlinearities

Take $\dot{x} = f(x) + Bu$ with $f(0) = 0$. One can consider this nonlinear system as a linear system with uncertainties, namely detach the nonlinear part:

$$\dot{x} = \mathrm{D}_0 f \cdot x + \gamma(x) + Bu \tag{1.15}$$

where $\mathrm{D}_0 f$ is the Jacobian of f at the origin and $\gamma(x) = f(x) - \mathrm{D}_0 f \cdot x$ is the uncertainty.

Like in the previous section, a state feedback control and a quadratic Lyapunov function is sought. A sufficient condition is

$$x^\top \begin{pmatrix} I \\ \mathrm{D}_0 f + BK \end{pmatrix}^\top \begin{pmatrix} 0 & P \\ P & 0 \end{pmatrix} \begin{pmatrix} I \\ \mathrm{D}_0 f + BK \end{pmatrix} x + 2x^\top P\gamma(x) < 0$$

for all $x \neq 0$. Of course, the nonlinear function γ has to obey some restrictions in order to control the above system.

Lure systems

Consider a nonlinear system of the following type (see e.g. [3]).

$$\dot{x} = Ax + G\gamma(z) + Bu \tag{1.16}$$

$$z = Hx \tag{1.17}$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, G \in \mathbb{R}^{n\times p}$ and $H \in \mathbb{R}^{p\times n}$ are given constant matrices, $\gamma : \mathbb{R}^p \mapsto \mathbb{R}^p$ is a given, nonlinear function, which satisfies so called *growth bounded sector condition*

$$\left(\underline{\beta} z - \gamma(z)\right)^\top \gamma(z) \geq 0 \tag{1.18}$$

with $\underline{\beta} = \mathrm{diag}\{\beta_1 \dots \beta_p\} \geq 0$.

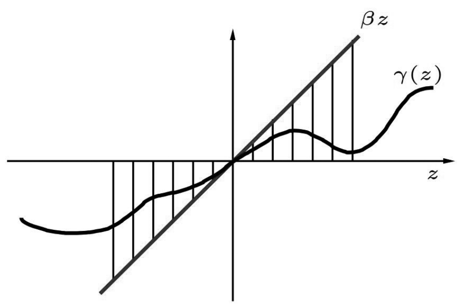


Figure 1.1: Sector condition of Lure systems, for $p = 1$, from [3]

This type of uncertainty will also appear in section 1.3.

1.2 Techniques and tools

In this section we state some well known theorems and tools, which are frequently used in the literature of control theory and also in this work.

1.2.1 Linear Matrix Inequality

Linear Matrix Inequality (LMI) is a convex optimization problem. Let $\mathbf{x} \in \mathbb{R}^n$ and $P(\mathbf{x})^\top = P(\mathbf{x}) \in \mathbb{R}^{m\times m}$ be a symmetric matrix *affinely depending* on the elements of $\mathbf{x}$ i. e. $P(\mathbf{x}) = P(x_1, \ldots, x_n) = A_0 + x_1 A_1 + \ldots + x_n A_n$ with $A_i{}^\top = A_i \in \mathbb{R}^{m\times m}$. The feasibility problem is to find a point in the set

$$\mathcal{F} = \{\mathbf{x} \in \mathbb{R}^n |\, P(\mathbf{x}) < 0\}\,. \tag{1.19}$$

One can easily check the convexity of the set $\mathcal{F}$. Let $\mathbf{x}^1, \mathbf{x}^2 \in \mathcal{F}$, then $P(\mathbf{x}^1) < 0$ and $P(\mathbf{x}^2) < 0$. It is sufficient to show that

$$\frac{1}{2}\left(P(\mathbf{x}^1) + P(\mathbf{x}^2)\right) < 0$$

which means that

$$\frac{1}{2}\left(A_0 + x_1^1 A_1 + \ldots + x_n^1 A_n + A_0 + x_1^2 A_1 + \ldots + x_n^2 A_n\right) < 0.$$

This is equivalent to

$$A_0 + \frac{x_1^1 + x_1^2}{2} A_1 + \ldots + \frac{x_n^1 + x_n^2}{2} A_n < 0$$

which is exactly $\frac{\mathbf{x}^1+\mathbf{x}^2}{2} \in \mathcal{F}$.

Supplementing the feasibility problem with a linear cost function turns it into a convex optimization problem, called semidefinite programming (SDP):

$$\begin{aligned} &\min\ \mathbf{c}^\top \cdot \mathbf{x} \\ &\text{subject to} \\ &P(\mathbf{x}) \le \mathbf{0} \end{aligned} \tag{1.20}$$

where $\mathbf{x}$ is the decision variable and $\mathbf{c} \in \mathbb{R}^n$ is given.

The LMI problem can be polynomially solved to arbitrary precision ϵ with interior point methods since the late 80's. The development and the implementation of this algorithm lasted almost a decade and had a great effect on the theory and application of optimization.[2]

1.2.2 Abstract multiplier method

In this section we discuss a certain type of matrix inequalities. In contrast to LMIs, there are problems, where the negativity is required on a given set, not on the whole linear space. If the set satisfies some regularity conditions, the problem can be reformulated as an LMI.

Suppose that we have a matrix $\Psi = \Psi^\top \in \mathbb{R}^{j\times j}$ which depends on some parameters in an affine way. As we have seen in Section 1.2.1, the matrix inequality

$$y^\top \Psi y < 0 \quad \forall y \in \mathbb{R}^j \setminus \{0\}$$

can be efficiently solved. If we require negativity for a given subset $\Omega \subset \mathbb{R}^j$, then the problem reads

$$\sup_{y\in\Omega} y^\top \Psi y < 0.$$

If the set Ω satisfy some regularity property, then the latter problem can be reduced to a matrix inequality. In [8], this method is presented and used for a broad class of robustness problems.

Suppose that $\mathcal{B} \subset \mathbb{R}^N$ is a linear subspace and matrices $U \in \mathbb{R}^{j\times N}$ and $V \in \mathbb{R}^{l\times N}$ are fixed, where V has maximum rank (complete row rank). Let $\mathcal{Q} \subset \mathbb{R}^l$ be given and assume that

$$V\mathcal{B} \cap \mathcal{Q} \neq \emptyset.$$

Definition 1.2.1 ([1])**.** *A symmetric matrix M is called a* multiplier matrix *for $\mathcal{Q}$ if $\xi^\top M\xi \geq 0$ for all $\xi \in \mathcal{Q}$.*

If this inequality is strict, then M is called a positive multiplier matrix *for $\mathcal{Q}$.*

Definition 1.2.2 ([1])**.** *The set $\mathcal{M}^+$ of positive multiplier matrices for $\mathcal{Q}$ is called a* sufficiently rich set of positive multipliers *for $\mathcal{Q}$, if for any positive multiplier $\overline{M}$ for $\mathcal{Q}$ there exists an element $M \in \mathcal{M}^+$ such that $M \leq \overline{M}$.*

Introduce the following set

$$\mathcal{B}_\mathcal{Q} = \{y \in \mathcal{B} : \; Vy \in \mathcal{Q}\}.$$

Consider the problem of solving inequalities of the following type

$$y^\top U^\top \Psi U y < 0 \quad \text{for all } y \in \mathcal{B}_\mathcal{Q}, y \neq 0. \tag{1.21}$$

Let $\mathcal{B}_0$ be a maximal subspace where the negativity fails:

$$\mathcal{B}_0 \subseteq \left\{x \in \mathcal{B} : \; x^\top U^\top \Psi U x \geq 0\right\} \subseteq \mathcal{B} \;\; \text{of maximal dimension.} \tag{1.22}$$

Assumption 1.

$$\mathcal{Q} \textit{ is a cone,} \tag{1.23}$$

$$\mathcal{B}_0 \cap \mathcal{B}_{\overline{\mathcal{Q}}} = \{0\} \quad \forall \mathcal{B}_0, \tag{1.24}$$

and

$$\textit{either } V\mathcal{B} \supset \mathcal{Q} \textit{ or } \mathcal{Q} \textit{ is closed.} \tag{1.25}$$

If the dimension of $\mathcal{B}_0$ is equal to zero, then $U^\top \Psi U < 0$ is satisfied on $\mathcal{B}$, thus there is an $\varepsilon > 0$ such that (1.21) equivalent to

$$y^\top (U^\top \Psi U + \varepsilon V^\top V) y < 0 \ \text{ for all } y \in \mathcal{B},\ y \neq 0.$$

Evidently, $\varepsilon I_l \in \mathcal{M}^+$.

In [8], it was proved that, if $\dim \mathcal{B}_0 \geq 1$ for some $\mathcal{B}_0$ and

$$\mathcal{B}_0 \cap \mathcal{B}_{\overline{\mathrm{cone}\mathcal{Q}}} \neq \{0\},$$

then the strict inequality

$$y^\top (U^\top \Psi U + V^\top M V) y < 0$$

is not satisfied for all $0 \neq y \in \mathcal{B}$ whatever multiplier matrix for $\mathcal{Q}$ is considered. Therefore condition (1.24) in Assumption 1 is not a technical one. However, one do not loose the generality with this condition.

Lemma 1.2.3 ([8])**.** *Assume that Assumption 1 holds true, and $\mathcal{M}^+$ is a sufficiently rich set of positive multipliers for $\mathcal{Q}$. Then the following statements are equivalent.*

1. *Inequality*

$$y^\top U^\top \Psi U y < 0$$

holds true for all $0 \neq y \in \mathcal{B}_{\mathcal{Q}}$ (this is (1.21)).

2. *There exists a $M \in \mathcal{M}^+$ such that*

$$y^\top (U^\top \Psi U + V^\top M V) y < 0, \quad y \in \mathcal{B},\ y \neq 0.$$

Remark 1.2.4. *If the set of positive multipliers for $\mathcal{Q}$ is not sufficiently rich then statement 2. is sufficient, however not necessary for 1. to hold.*

1.2.3 Linearization Lemma

The following Lemma is essentially used to transform nonlinear matrix inequalities into linear ones. Matrix inequalities (linear and nonlinear ones) occur frequently in control theory, but for computational reasons we aim to derive LMIs whenever it is possible.

Lemma 1.2.5 ([13]). *Suppose that A and S are constant matrices, that $B(v), Q(v) = Q(v)^\top$ depend affinely on a parameter v, and that $R(v)$ can be decomposed as $R(v) = T \cdot U(v)^{-1} \cdot T^\top$ with $U(v)$ being affine.*
Then the nonlinear matrix inequalities

$$U(v) > 0, \qquad \begin{pmatrix} A \\ B(v) \end{pmatrix}^\top \cdot \begin{pmatrix} Q(v) & S \\ S^\top & R(v) \end{pmatrix} \cdot \begin{pmatrix} A \\ B(v) \end{pmatrix} < 0 \tag{1.26}$$

are equivalent to the linear matrix inequality

$$\left(\begin{array}{c|c} A^\top Q(v) A + A^\top S B(v) + B(v)^\top S^\top A & B(v)^\top T \\ \hline T^\top B(v) & -U(v) \end{array}\right) < 0 \tag{1.27}$$

.

Note that v is just for emphasise the dependencies. One can think of a parameter-dependent matrix as a matrix with some unknown entries, these are the variables.

We remark that the statement can easily be deduced with Schur complement.

1.3 Nonlinear Model Predictive Control

In this section we introduce the Nonlinear Model Predictive Control (NMPC) technique from [3]. A comprehensive overview of the method can be found in [12].

System setup

The system is a Lure system, described by (1.16)-(1.17) with uncertainty condition (1.18).

Control constraints

A control u is constrained by the polytope

$$\mathcal{C} = \left\{ \begin{pmatrix} x \\ u \end{pmatrix} \in \mathbb{R}^{n+m} \;:\; c_i x + d_i u \leq 1, i = 1 \dots r \right\} \tag{1.28}$$

with given (row) vectors $c_i \in \mathbb{R}^{1\times n}, d_i \in \mathbb{R}^{1\times m}$. For every $u(t)$ to be applied, $\left(\begin{smallmatrix} x(t) \\ u(t) \end{smallmatrix}\right) \in \mathcal{C}$ is required along the solution $x(t)$.

Control task

Let $0 = t_0 < t_1 < \ldots < t_k < \ldots$ be a sequence of sampling instants. For every time instant t_k a state feedback $K_k \in \mathbb{R}^{m\times n}$ will be calculated and $u(t) = K_k x(t)$ will be applied at time interval $t \in [t_k, t_{k+1})$. A cost functional (cost-to-go) is given with positive definite matrices $Q \in \mathbb{R}^{n\times n}, R \in \mathbb{R}^{m\times m}$.

$$J(x(\cdot), u(\cdot)) = \int_{t_k}^{\infty} \|x(\tau)\|_Q^2 + \|u(\tau)\|_R^2 \mathrm{d}\tau \tag{1.29}$$

The task is to find matrices K_k such that the corresponding state-feedback control stabilizes the system, fulfills the control constraints and the cost $J_0(x(\cdot), u(\cdot))$ is bounded. In this case, we shall talk about a cost guaranteeing control.

Lemma 1.3.1 ([3])**.** *Consider the system (1.16)-(1.17) with uncertainty condition (1.18). NMPC controller is designed by the repeated solution of the optimization problem*

$$\min_{\alpha_k, \tau, \Lambda_k, \Gamma_k} \alpha_k \tag{1.30}$$

$$\textit{subject to}$$

$$\begin{pmatrix} 1 & x(t_k)^\top \\ x(t_k) & \Lambda_k \end{pmatrix} > 0 \tag{1.31}$$

$$\begin{pmatrix} -\Delta_k - \Delta_k^\top & -S_k & \Lambda_k Q^{\frac{1}{2}} & \Gamma_k^\top R^{\frac{1}{2}} \\ -S_k^\top & \alpha_k \tau I & 0 & 0 \\ Q^{\frac{1}{2}}\Lambda_k & 0 & \alpha_k I & 0 \\ R^{\frac{1}{2}}\Gamma_k & 0 & 0 & \alpha_k I \end{pmatrix} > 0 \tag{1.32}$$

$$\begin{pmatrix} 1 & c_i\Lambda_k + d_i\Gamma_k \\ (c_i\Lambda_k + d_i\Gamma_k)^\top & \Lambda_k \end{pmatrix} \geq 0 \tag{1.33}$$

$$i = 1, \ldots, r$$

at the sampling instant t_k based on the state $x(t_k)$, where

$$\Delta_k = A \cdot \Lambda_k + B \cdot \Gamma_k \quad \textit{and}$$
$$S_k = G\alpha_k + \frac{\tau}{2}\Lambda_k H^\top \beta^\top.$$

The problem, with $P_k = \alpha_k \Lambda_k^{-1}$ and $K_k = \Gamma_k \Lambda_k^{-1}$, has the following properties:

1. *The optimization problem is an LMI, aside from τ. Furthermore it is feasible at the sampling instant t_{k+1} if it is feasible at t_k.*

2. *The solution to the optimization problem minimizes the upper bound $\overline{V}_k = x(t_k)^\top P_k x(t_k)$ on the cost functional (1.29) at each sampling instant t_k.*

3. *If the optimization problem is feasible at $t_0 = 0$, the control law*

   $$u(t) = K_k x(t), \;\; t \in [t_k, t_{k+1})$$

   asymptotically stabilizes the origin of the system (1.16)-(1.18) and the control constraints (1.28) are satisfied for time $t \geq 0$.

In [3] it is claimed, that τ is considered fixed and not an optimization variable. A reasonable fixed value for τ can be determined off-line, the resulting performance is only weakly sensitive towards it.

The above described result is invariant in time, precisely, if K_0 would be applied on $[0, \infty)$ then the stability and the cost are also guaranteed. However one wish to improve the performance by re-calculating K_k at each time instant.

The matrix inequalities can be categorized into three groups.

- assuring stability (via the Lyapunov function P) (1.32)
- assuring control constraints (1.33)
- assuring the coherence between the time intervals (1.31)

The un-updated control K_0 applied on $[0, \infty)$ would also satisfy the control task. However if one wish to improve the control as time evolves, then recalculation and (1.31) is needed. The inequality (1.33) is needed to the control constraints, independently on the sampling instants.

Chapter 2

Main Result

In the following section we introduce a certain class of uncertain systems. Then we propose a stabilizing, cost guaranteeing method with control constraints. In the further sections the validity of the method is proven and some demonstrating applications are presented.

2.1 Problem statement

The system

Consider the system

$$\dot{x} = Ax + Bu + Ew + Hp, \tag{2.1}$$

$$y = Cx, \tag{2.2}$$

$$\zeta = \begin{pmatrix} C_\zeta x \\ D_\zeta u \end{pmatrix}, \tag{2.3}$$

$$q_i = A_{qi}x + B_{qi}u + G_i p_i, \tag{2.4}$$

$$i = 1 \dots s$$

where $x \in \mathbb{R}^{n_x}$ is the state, $u \in \mathbb{R}^{n_u}$ is the control and $w : \mathbb{R}_+ \mapsto \mathbb{R}^{n_w}$ is the exogenous disturbance. The disturbance is an external time-dependent function which perturbs our system, it is not explicitly known, but we will impose some conditions/bounds later. The uncertainty/nonlinearity appears in $p \in \mathbb{R}^{l_p}$ and it is partitioned as

$$p = \begin{pmatrix} p_1 \\ \vdots \\ p_s \end{pmatrix} \in \mathbb{R}^{l_p}$$

where $0 \leq s$, $\sum_{i=1}^{s} l_{p_i} = l_p$. The uncertainties will be specified later. Furthermore, $q^\top = \left(q_1^\top, \ldots, q_s^\top\right)$ is the uncertain output where $q_i \in \mathbb{R}^{l_{q_i}}$ and $\sum_{i=1}^{s} l_{q_i} = l_q$.

The measured output is $y \in \mathbb{R}^{n_y}$, this means that the state space is only available via y. In many applications the matrix C has the following form.

$$C = \begin{pmatrix} 1 \quad 0 \quad \cdots & & \\ & 1 \quad 0 & \\ & & 1 \quad 0 \quad \ldots \end{pmatrix}$$

This means that some coordinates are unmeasurable e.g. the positions are known, but the velocities not.

We call $\zeta \in \mathbb{R}^{n_x + x_u}$ the penalty output, and D_ζ is assumed to be nonsingular.

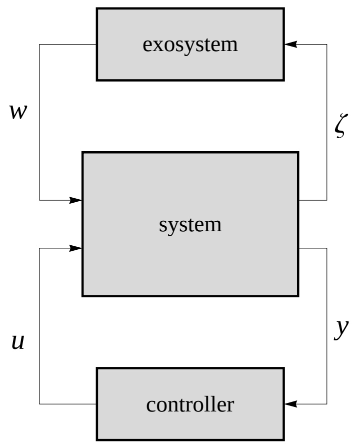


Figure 2.1: Schematic picture of system (2.1)-(2.3)

Uncertainty constraints

All nonlinearities/uncertainties are represented by functions p_i possibly depending on t, x and u. The only available information about p and q is that their values are constrained by a given set $\Omega \subset \mathbb{R}^{l_p + l_q}$, which is defined by quadratic constrains:

$$\Omega = \left\{ \begin{pmatrix} p \\ q \end{pmatrix} \in \mathbb{R}^{l_p + l_q} \,\middle|\, \begin{pmatrix} p_i \\ q_i \end{pmatrix}^\top \begin{pmatrix} Q_{0i} & S_{0i} \\ S_{0i}^\top & R_{0i} \end{pmatrix} \begin{pmatrix} p_i \\ q_i \end{pmatrix} \geq 0, i = 1 \ldots s \right\} \tag{2.5}$$

where $Q_{0i} = Q_{0i}^\top$, $R_{0i} = R_{0i}^\top \geq 0$ and S_{0i} are given, constant matrices of appropriate dimension. This partitioning allows multiple uncertainties (such as e.g. nonlinearities, parametric uncertainties). We shall use the notation

$$Q_0 = \operatorname{diag}\{Q_{01}, \ldots, Q_{0s}\}$$
$$R_0 = \operatorname{diag}\{R_{01}, \ldots, R_{0s}\}$$
$$S_0 = \operatorname{diag}\{S_{01}, \ldots, S_{0s}\}.$$

This formulation covers several frequently investigated types of uncertainties with $s = 1$. If $Q_0 = 0$, $S_0 = I$ and $R_0 = 0$, then one speaks about positive real uncertainty, if $Q_0 = -I$, $S_0 = 0$ and $R_0 = I$, then one has norm bounded uncertainties, and if $Q_0 = \frac{1}{2}(K_1^\top K_2 + K_2^\top K_1)$, $S_0 = \frac{1}{2}(K_1 + K_2)^\top$ and $R_0 = I$, then one faces the case of sector-bounded uncertainties.

These matrices are assumed to satisfy the following conditions.

Assumption 2. *Inequalities*

$$R_0 \geq 0 \tag{2.6}$$

and

$$Q_0 + \mathcal{G}^\top S_0^\top + S_0 \mathcal{G} + \mathcal{G}^\top R_0 \mathcal{G} < 0 \tag{2.7}$$

hold true with $\mathcal{G} = \operatorname{diag}\{G_1, \ldots, G_s\} \in \mathbb{R}^{l_q \times l_p}$.

We note that the positive semi-definiteness of R_0 assures that the system (2.1)-(2.4) is well posed, i.e. for any (x, u) there exists p such that $\left(p^\top, q^\top\right)^\top \in \Omega$. Condition (2.7) of Assumption 2 implies that $\left(p^\top, p^\top \mathcal{G}^\top\right)^\top \in \Omega$ if and only if $p = 0$, thus the origin is an equilibrium point of the unperturbed, uncertain/nonlinear system. Moreover, the set of uncertain input vectors satisfying $(p^\top, q^\top)^\top \in \Omega$ is bounded if q is defined by (2.4) and (x, u) comes from a bounded set, which is also a reasonable requirement.

Control constraints

The control is quadratically constrained, i.e.

$$u^\top Q_u u = \|u\|_{Q_u}^2 \leq 1 \tag{2.8}$$

must be satisfied for a given matrix $Q_u = Q_u^\top \geq 0$. If no control constraint is required, then one can assign $Q_u = 0$.

State constraints

The state is not directly known, but it is assumed that the initial condition is explicitly known

$$x(0) = x_0.$$

The method can be extended, when the initial state is known up to the norm bound

$$\|x\|^2 \leq \rho$$

which is a weaker restriction, see [10].

Constraints on disturbances

The disturbances to be investigated are restricted to one of the following classes.

Definition 2.1.1 (Class Δ_I)**.** *The disturbances are produced by an exosystem, the input of which is the penalty output ζ of the original system (2.1)-(2.3), the output is w, and (ζ, w) satisfy the inequality*

$$\|w\|_{S_L}^2 \leq \gamma_\Delta \|\zeta\|^2 \tag{2.9}$$

with some $0 \leq \gamma_\Delta \leq 1$.

The above inequality means that the exosystem does not operate as an amplifier; its output is bounded by the penalty output ζ.

Definition 2.1.2 (Class Δ_{II})**.** *The function $w : \mathbb{R} \mapsto \mathbb{R}^{n_w}$ is in $\mathcal{L}_2(\mathbb{R}, \mathbb{R}^{n_w})$ and*

$$\int_0^\infty \|w(t)\|_{S_L}^2 \mathrm{d}t \leq \eta \tag{2.10}$$

for some given positive constant η.

This means that the disturbances have finite energy.
We will prove different convergence properties for the different classes.

Control task

The control task is to construct a dynamic output feedback controller which asymptotically stabilizes a certain neighbourhood of the origin of (2.1). For the solution of this task, we propose to use a variant of NMPC. Let $0 = t_0 < t_1 < \ldots < t_k < \ldots$ denote the sequences of sampling instants. At each sampling instant t_k the cost function

$$J_k(x(\cdot), u(\cdot), w(\cdot)) = \int_{t_k}^\infty L(x(t), u(t), w(t))\mathrm{d}t \tag{2.11}$$

is assigned where

$$L(x, u, w) = x^\top Q_L x + u^\top R_L u - w^\top S_L w \overset{(2.3)}{=} \|\zeta\|^2 - \|w\|_{S_L}^2$$

with $0 < S_L{}^\top = S_L \in \mathbb{R}^{n_w \times n_w}$ given and $Q_L = C_\zeta^\top C_\zeta$, $R_L = D_\zeta^\top D_\zeta$. Thus, it follows that Q_L, R_L and S_L are symmetric, Q_L is positive semidefinite, R_L and S_L are positive definite matrices.

We look for the controller in the following form:

$$\dot{\hat{x}} = A_c^k \hat{x} + L_c^k y, \qquad \hat{x}(t_k) = \hat{x}_k, \hat{x}_0 = 0 \tag{2.12}$$

$$u = K_c^k \hat{x} \tag{2.13}$$

on $[t_k, t_k + 1)$

where $\hat{x} \in \mathbb{R}^{n_x}$. The unknown matrices A_c^k, L_c^k, K_c^k vary with k, but we omit the index k if it does not cause any confusion. In receding horizon feedback framework, the idea is to find a controller (or a control policy) that gives the minimal, or minimax value of the cost function. Being the system uncertain, the computation of the minimal (or minimax) value would be an excessive task. Instead, one should be satisfied at each sampling instant with an upper bound of this minimal value, in other words, with a guaranteed cost controller.

2.2 Deriving the inequalities

In this section we derive certain matrix inequalities to find the matrices in the above control task. The inequalities will be partitioned into three types, like in Lemma 1.3.1.

In the next section it will be proven that the matrices, found with these inequalities, indeed satisfy the control task.

2.2.1 The main LMI

In this section a Lyapunov function is sought. We do not concern the control constraints and the sampling instants yet.

The closed loop system

Let

$$z := \begin{pmatrix} x \\ \hat{x} \end{pmatrix} \in \mathbb{R}^{2n_x} \tag{2.14}$$

$$\mathcal{A} := \left(\begin{array}{c|c} A & BK_c \\ \hline L_cC & A_c \end{array}\right) \in \mathbb{R}^{2n_x \times 2n_x} \tag{2.15}$$

$$\mathcal{E} := \begin{pmatrix} E \\ 0 \end{pmatrix} \in \mathbb{R}^{2n_x \times n_w} \tag{2.16}$$

$$\mathcal{H} := \begin{pmatrix} H \\ 0 \end{pmatrix} \in \mathbb{R}^{2n_x \times l_p} \tag{2.17}$$

$$\mathcal{A}_q := \left(\begin{array}{c|c} A_{q1} & B_{q1}K_c \\ \vdots & \vdots \\ A_{qs} & B_{qs}K_c \end{array}\right) \in \mathbb{R}^{l_q \times 2n_x} \tag{2.18}$$

then (2.1)-(2.2) is equivalent to

$$\dot{z} = \mathcal{A}z + \mathcal{E}w + \mathcal{H}p \tag{2.19}$$

$$q = \mathcal{A}_q z + \mathcal{G}p \tag{2.20}$$

and the running cost in the cost function (2.11) can be represented as

$$L(x,u,w) = \mathcal{L}(z,w) = (*)\overbrace{\begin{pmatrix} Q_L & & \\ & R_L & \\ & & -S_L \end{pmatrix}}^{\Xi:=}\begin{pmatrix} I & & \\ & K_c & \\ & & I \end{pmatrix}\begin{pmatrix} x \\ \hat{x} \\ w \end{pmatrix}. \tag{2.21}$$

Guaranteed cost

Definition 2.2.1. *Consider the nonlinear/uncertain system*

$$\dot{z} = f(z,u,w,p)$$
$$q = g(z,u,p)$$

with cost function of the type (2.11) and with a given set of nonlinearities or uncertainties Ω *i. e.* $\left(p^\top, q^\top\right)^\top \in \Omega$.
The state-feedback control $u = k(z)$ *is a* guaranteeing cost robust minimax strategy *with a* decay rate $\delta \geq 0$ *if there exists a positive definite function* $\mathcal{V} : \mathbb{R}^{n_z} \to \mathbb{R}^+$ *such that*

$$\sup_{\binom{p}{q}\in\Omega} \{\nabla\mathcal{V}^\top(z) f(z,k(z),w,p) + L(z,k(z),w)\} \leq -\delta\mathcal{V}(z) \tag{2.22}$$

holds for all z *and* w, $\left(z^\top, w^\top\right) \neq \left(0^\top, 0^\top\right)$.

Note that the purpose of $\delta > 0$ is the uniform decay rate. δ is considered to be a given constant, it can be set to zero if there is no need to control the decay rate.

We will search $\mathcal{V}$ in the special quadratic form $\mathcal{V}(z) = z^\top P z$ ($P^\top = P \in \mathbb{R}^{2n_x \times 2n_x}$). Therefore $\nabla\mathcal{V}(z) = 2Pz$ and the first term in (2.22) becomes

$$\nabla\mathcal{V}^\top(z)\underbrace{f(z,k(z),w,p)}_{f(z,w,p)} = 2z^\top \cdot P \cdot f(z,w,p) =$$
$$= \left(z^\top, f(z,w,p)^\top\right)\underbrace{\begin{pmatrix} 0 & P \\ P & 0 \end{pmatrix}}_{\Phi\otimes P}\begin{pmatrix} z \\ f(z,w,p) \end{pmatrix}$$

where $\Phi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathbb{R}^{2\times 2}$. P is a variable to be determined, we will include it in the inequalities.

Note that the use of Φ is unnecessary, however we want to point out that the calculations can be adapted to discrete time systems. The system derivative and the Lyapunov technique is slightly different, but the proof can be generalized by altering the matrix Φ, see [9].

In our system (2.19)-(2.20), with uncertainty constraints (2.5) and cost function (2.21), inequality (2.22) takes the following form

$$\sup_{\binom{p}{q}\in\Omega} F(z,w,p) \leq -\delta z^\top P z \tag{2.23}$$

$$\text{where } F(z,w,p) =$$

$$(*)(\Phi\otimes P)\begin{pmatrix} I & 0 & 0 \\ \mathcal{A} & \mathcal{E} & \mathcal{H}\end{pmatrix}\begin{pmatrix} z \\ w \\ p\end{pmatrix} + (*)\Xi\begin{pmatrix} \begin{pmatrix} I & 0 \\ 0 & K_c\end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0\end{matrix} \\ \begin{pmatrix} 0 & 0\end{pmatrix} & \begin{matrix} I & 0\end{matrix}\end{pmatrix}\begin{pmatrix} z \\ w \\ p\end{pmatrix}.$$

Abstract multipliers

Note that inequality (2.23) is not a matrix inequality, since the positivity is required only on a subset, not on the whole space. To transform it into a matrix inequality, we will apply the abstract multiplier method described in section 1.2.2.

First let

$$\Psi = \Psi^\top = \operatorname{diag}\left\{\underbrace{\begin{pmatrix} \delta\cdot P & P \\ P & 0\end{pmatrix}}_{4n_x}, \underbrace{\Xi}_{n_x+n_u+n_w}, \underbrace{0}_{l_p+l_q}\right\}, \tag{2.24}$$

$$V = \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I\end{pmatrix}}_{\in\mathbb{R}^{(4n_x+n_x+n_u+n_w+l_p+l_q)\times(l_p+l_q)}}, \quad \mathcal{L}_1 = \begin{pmatrix} I & 0 \\ \mathcal{A} & \mathcal{E} \\ \begin{pmatrix} I & 0 \\ 0 & K_c\end{pmatrix} & 0 \\ 0 & I \\ 0 & 0 \\ \mathcal{A}_q & 0\end{pmatrix}, \quad \mathcal{L}_0 = \begin{pmatrix} 0 \\ \mathcal{H} \\ 0 \\ 0 \\ I \\ \mathcal{G}\end{pmatrix} \tag{2.25}$$

and

$$\mathcal{B}_1 := \operatorname{Im}\mathcal{L}_1,\ \mathcal{B}_0 := \operatorname{Im}\mathcal{L}_0 \tag{2.26}$$

$$\mathcal{B} := \mathcal{B}_1 \oplus \mathcal{B}_0. \tag{2.27}$$

Note that $\mathcal{B}_1 \cap \mathcal{B}_0 = \{0\}$.

With these matrices we can reformulate (2.23). Mind the following product

$$
\begin{array}{ccccc}
 & & \ast & \begin{array}{|c|}\hline z \\ w \\ p \\ \hline \end{array} & \\
\begin{array}{|cc|c|}\hline
I & 0 & 0 \\
\mathcal{A} & \mathcal{E} & \mathcal{H} \\
\begin{pmatrix} I & 0 \\ 0 & K_c \end{pmatrix} & 0 & 0 \\
0 & I & 0 \\
0 & 0 & I \\
\mathcal{A}_q & 0 & \mathcal{G} \\ \hline
\end{array} & & & = & \begin{array}{|c|}\hline z \\ \dot{z} \\ \begin{pmatrix} x \\ u \end{pmatrix} \\ w \\ p \\ q \\ \hline \end{array} \\
\mathcal{L}_1 \qquad\quad \mathcal{L}_0 & & & & y
\end{array}
$$

where $y \in \mathcal{B}$ and $Vy = \left(\begin{smallmatrix} p \\ q \end{smallmatrix}\right)$. Let $\mathcal{B}_\Omega := \{y \in \mathcal{B} : Vy \in \Omega\}$. Then $y \in \mathcal{B}_\Omega$ is equivalent to $\left(\begin{smallmatrix} p \\ q \end{smallmatrix}\right) \in \Omega$.

Now we can see that (2.23) is equivalent to

$$y^\top \Psi y < 0 \quad \forall y \in \mathcal{B}_\Omega. \tag{2.28}$$

and we are in the setup of section 1.2.2 with $N = 5n_x + n_u + n_w + l_p + l_q$, $\mathcal{Q} = \Omega$ and $U = I_N$. Notice that (2.28) is equivalent to

$$y^\top (\alpha\Psi) y < 0 \quad \forall y \in \mathcal{B}_\Omega \tag{2.29}$$

for any $\alpha > 0$.

$$\alpha\Psi = \alpha \operatorname{diag}\left\{\begin{pmatrix} \delta \cdot P & P \\ P & 0 \end{pmatrix}, \Xi, 0\right\} = \operatorname{diag}\left\{\begin{pmatrix} \alpha\delta P & \alpha P \\ \alpha P & 0 \end{pmatrix}, \alpha\Xi, 0\right\} \tag{2.30}$$

Since P is a variable and we have given constant matrices in Ξ, we simply call $P_\alpha := \alpha P$ a new variable and Q_L, R_L, S_L alter with a multiplier α.

The α multiplier gives us an additional degree of freedom in the decision variables.

Lemma 2.2.2 ([9])**.** *For positive constants $\tau_i, \epsilon_i > 0, i = 1 \dots s$ let*

$$
\begin{aligned}
\underline{\tau} &= \operatorname{diag}\left\{\tau_1 I_{l_{p_1}}, \dots, \tau_s I_{l_{p_s}}\right\} \\
\underline{\underline{\tau}} &= \operatorname{diag}\left\{\tau_1 I_{l_{q_1}}, \dots, \tau_s I_{l_{q_s}}\right\} \\
\underline{\varepsilon} &= \operatorname{diag}\left\{\varepsilon_1 I_{l_{p_1}}, \dots, \varepsilon_s I_{l_{p_s}}\right\} \\
\underline{\underline{\varepsilon}} &= \operatorname{diag}\left\{\varepsilon_1 I_{l_{q_1}}, \dots, \varepsilon_s I_{l_{q_s}}\right\}.
\end{aligned}
$$

The set

$$\mathcal{M}^+ = \left\{ M = \begin{pmatrix} \underline{\tau} Q_0 + \underline{\varepsilon} & \underline{\tau} S_0 \\ S_0^\top \underline{\tau} & \underline{\underline{\tau}} R_0 + \underline{\underline{\varepsilon}} \end{pmatrix} \middle| \, \tau_i, \varepsilon_i > 0, \; i = 1, \dots, s \right\} \tag{2.31}$$

consists of positive multiplier matrices for Ω. If $s = 1$, then $\mathcal{M}^+$ is sufficiently rich.

We remark that Assumption 2 in the uncertainty constraints is necessary in Lemma 2.2.2.

Hence Lemma 1.2.3 can be applied and the guaranteed cost problem is reduced to finding matrices P, A_c, K_c, L_c, M (described above) and positive constants $\alpha, \tau_i, \epsilon_i$ such that

$$\Psi + V^\top M V < 0 \quad \forall y \in \mathcal{B}$$

which is equivalent to

$$(*) \operatorname{diag}\left\{ \begin{pmatrix} \alpha\delta P & \alpha P \\ \alpha P & 0 \end{pmatrix}, \alpha\Xi, M \right\} (\mathcal{L}_1, \mathcal{L}_0) < 0. \tag{2.32}$$

Note that for $s = 1$ the inequality (2.32) is an equivalent condition of the existence of the quadratic Lyapunov function. If $s > 1$ then the condition is sufficient.

Linearization lemma and Schur complements

In this section we reformulate the nonlinear inequality (2.32), which is:

$$(*) \begin{pmatrix} \delta P_\alpha & P_\alpha & 0 & 0 & 0 & 0 & 0 \\ P_\alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha Q_L & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha R_L & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha S_L & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \underline{\tau} Q_0 + \underline{\varepsilon} & \underline{\tau} S_0 \\ 0 & 0 & 0 & 0 & 0 & S_0^\top \underline{\tau} & \underline{\underline{\tau}} R_0 + \underline{\underline{\varepsilon}} \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ \mathcal{A} & \mathcal{E} & \mathcal{H} \\ \begin{pmatrix} I & 0 \\ 0 & K_c \end{pmatrix} & 0 & 0 \\ \begin{pmatrix} I & 0 \\ 0 & K_c \end{pmatrix} & 0 & 0 \\ 0 & I_{l_w} & 0 \\ 0 & 0 & I_{l_p} \\ \mathcal{A}_q & 0 & \mathcal{G} \end{pmatrix} < 0.$$

Now exchange the subspaces by multiplying the middle block-diagonal matrix from the right by $L^\top L = I$ and from the left by and $L \cdot L^\top = I$ where

$$L = \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 \end{pmatrix}.$$

This yields:

$$
(*)\left(\begin{array}{cccc|ccc}
\delta P_\alpha & P_\alpha & 0 & 0 & 0 & 0 & 0 \\
P_\alpha & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \underline{\tau} Q_0 + \underline{\varepsilon} & 0 & \underline{\tau} S_0 & 0 & 0 \\
0 & 0 & 0 & -\alpha S_L & 0 & 0 & 0 \\
\hline
0 & 0 & S_0^\top \underline{\tau} & 0 & \underline{\underline{\tau}} R_0 + \underline{\underline{\varepsilon}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \alpha Q_L & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \alpha R_L
\end{array}\right)
\left(\begin{array}{ccc}
I & 0 & 0 \\
\mathcal{A} & \mathcal{E} & \mathcal{H} \\
0 & 0 & I_{l_p} \\
0 & I_{l_w} & 0 \\
\hline
\mathcal{A}_q & 0 & \mathcal{G} \\
\multirow{2}{*}{$\begin{pmatrix} I & 0 \\ 0 & K_c \end{pmatrix}$} & 0 & 0 \\
 & 0 & 0
\end{array}\right) < 0.
$$

Note that diag $\left\{\underline{\underline{\tau}} R_0 + \underline{\underline{\varepsilon}}, \alpha Q_L, \alpha R_L\right\}$ is positive definite therefore the Linearization Lemma in section 1.2.3 (partitioned according to the solid lines) can be applied to obtain

$$
\left(\begin{array}{ccc|ccc}
\phi_{11} & * & * & & & \\
\mathcal{E}^\top P_\alpha & -\alpha S_L & * & & * & \\
\mathcal{H}^\top P_\alpha + \underline{\tau} S_0 \mathcal{A}_q & 0 & \substack{\underline{\tau} Q_0 + \underline{\varepsilon} + \\ \underline{\tau} S_0 \mathcal{G} + \mathcal{G}^\top S_0^\top \underline{\tau}} & & & \\
\hline
\mathcal{A}_q & 0 & \mathcal{G} & \frac{-1}{\alpha}\left(\underline{\underline{\tau}} R_0 + \underline{\underline{\varepsilon}}\right)^{-1} & 0 & 0 \\
\multirow{2}{*}{$\begin{pmatrix} I & 0 \\ 0 & K_c \end{pmatrix}$} & 0 & 0 & 0 & \frac{-1}{\alpha} Q_L^{-1} & 0 \\
 & 0 & 0 & 0 & 0 & \frac{-1}{\alpha} R_L^{-1}
\end{array}\right) < 0
$$

where $\phi_{11} = \delta P_\alpha + \mathcal{A}^\top P_\alpha + P_\alpha \mathcal{A}$. Let us use Schur complement, to remove $\frac{-1}{\alpha}\left(\begin{smallmatrix} Q_L & 0 \\ 0 & R_L \end{smallmatrix}\right)^{-1}$ from the lower-right block. Then we obtain

$$
\left(\begin{array}{ccc|c}
\phi'_{11} & * & * & \\
\mathcal{E}^\top P_\alpha & -\alpha S_L & * & * \\
\mathcal{H}^\top P_\alpha + \underline{\tau} S_0 \mathcal{A}_q & 0 & \substack{\underline{\tau} Q_0 + \underline{\varepsilon} + \\ \underline{\tau} S_0 \mathcal{G} + \mathcal{G}^\top S_0^\top \underline{\tau}} & \\
\hline
\mathcal{A}_q & 0 & \mathcal{G} & -\left(\underline{\underline{\tau}} R_0 + \underline{\underline{\varepsilon}}\right)^{-1}
\end{array}\right) < 0.
$$

with $\phi_{11}' = \delta P_\alpha + \alpha\overline{Q} + \mathcal{A}^\top P_\alpha + P_\alpha \mathcal{A}$, and $\overline{Q} := \left(\begin{smallmatrix} Q_L & 0 \\ 0 & K_c^\top R_L K_c \end{smallmatrix}\right)$. Now we move the bottom-right block away with a similar technique:

$$
\begin{aligned}
&\begin{pmatrix} \phi_{11}' & * & * \\ \mathcal{E}^\top P_\alpha & -\alpha S_L & * \\ \mathcal{H}^\top P_\alpha + \underline{\tau} S_0 \mathcal{A}_q & 0 & \underline{\tau} Q_0 + \underline{\tau} S_0 \mathcal{G} + \mathcal{G}^\top S_0^\top \underline{\tau} \end{pmatrix} + \\
&\begin{pmatrix} 0 \\ 0 \\ I \end{pmatrix} \underline{\varepsilon} \begin{pmatrix} 0 & 0 & I \end{pmatrix} + \begin{pmatrix} \mathcal{A}_q^\top \\ 0 \\ \mathcal{G}^\top \end{pmatrix} \left(\underline{\underline{\tau}} R_0 + \underline{\underline{\varepsilon}} \right) \begin{pmatrix} \mathcal{A}_q & 0 & \mathcal{G} \end{pmatrix} = \\
&\begin{pmatrix} \phi_{11}' & * & * \\ \mathcal{E}^\top P_\alpha & -\alpha S_L & * \\ \mathcal{H}^\top P_\alpha + \underline{\tau} S_0 \mathcal{A}_q & 0 & \underline{\tau} Q_0 + \underline{\tau} S_0 \mathcal{G} + \mathcal{G}^\top S_0^\top \underline{\tau} \end{pmatrix} + \\
&(*) \begin{pmatrix} \underline{\varepsilon} & 0 & 0 \\ 0 & \underline{\underline{\varepsilon}} & 0 \\ 0 & 0 & \underline{\underline{\tau}} \end{pmatrix} \begin{pmatrix} 0 & 0 & I \\ \mathcal{A}_q & 0 & \mathcal{G} \\ R_0^{\frac{1}{2}} \mathcal{A}_q & 0 & R_0^{\frac{1}{2}} \mathcal{G} \end{pmatrix}
\end{aligned}
$$

This time, we use the Schur complement in the other direction and obtain:

$$
\begin{pmatrix} \phi_{11}' & * & * & * & * & * \\ \mathcal{E}^\top P_\alpha & -\alpha S_L & * & * & * & * \\ \mathcal{H}^\top P_\alpha + \underline{\tau} S_0 \mathcal{A}_q & 0 & \substack{\underline{\tau} Q_0 + \\ \underline{\tau} S_0 \mathcal{G} + \mathcal{G}^\top S_0^\top \underline{\tau}} & * & * & * \\ 0 & 0 & I & -\underline{\varepsilon}^{-1} & 0 & 0 \\ \mathcal{A}_q & 0 & \mathcal{G} & 0 & -\underline{\underline{\varepsilon}}^{-1} & 0 \\ R_0^{\frac{1}{2}} \mathcal{A}_q & 0 & R_0^{\frac{1}{2}} \mathcal{G} & 0 & 0 & -\underline{\underline{\tau}}^{-1} \end{pmatrix} < 0
$$

Let us move back the matrix $\overline{Q}$ from ϕ_{11}' to the bottom-right corner and recall the matrices C_ζ, D_ζ.

$$
\begin{pmatrix} \phi_{11} & * & * & * & * & * & \left(\begin{smallmatrix} C_\zeta & 0 \\ 0 & D_\zeta K_c \end{smallmatrix}\right)^\top \\ \mathcal{E}^\top P_\alpha & -\alpha S_L & * & * & * & * & 0 \\ \mathcal{H}^\top P_\alpha + \underline{\tau} S_0 \mathcal{A}_q & 0 & \substack{\underline{\tau} Q_0 + \\ \underline{\tau} S_0 \mathcal{G} + \mathcal{G}^\top S_0^\top \underline{\tau}} & * & * & * & 0 \\ 0 & 0 & I & -\underline{\varepsilon}^{-1} & 0 & 0 & 0 \\ \mathcal{A}_q & 0 & \mathcal{G} & 0 & -\underline{\underline{\varepsilon}}^{-1} & 0 & 0 \\ R_0^{\frac{1}{2}} \mathcal{A}_q & 0 & R_0^{\frac{1}{2}} \mathcal{G} & 0 & 0 & -\underline{\underline{\tau}}^{-1} & 0 \\ \left(\begin{smallmatrix} C_\zeta & 0 \\ 0 & D_\zeta K_c \end{smallmatrix}\right) & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\alpha} \end{pmatrix} < 0.
$$

Let us use the congruence transformation with diag $\{P_\alpha^{-1}, I, \underline{\mathcal{T}}^{-1}, I, \dots I\}$. Only the first three columns are presented.

$$\begin{pmatrix} P_\alpha^{-1}\mathcal{A}^\top + \mathcal{A}P_\alpha^{-1} + \delta P_\alpha^{-1} & * & * & \cdots \\ \mathcal{E}^\top & -\alpha S_L & * & \cdots \\ \underline{\mathcal{T}}^{-1}\mathcal{H}^\top + S_0\mathcal{A}_q P_\alpha^{-1} & 0 & \substack{Q_0\underline{\mathcal{T}}^{-1} + \\ S_0\mathcal{G}\underline{\mathcal{T}}^{-1} + \underline{\mathcal{T}}^{-1}\mathcal{G}^\top S_0^\top} & \cdots \\ 0 & 0 & \underline{\mathcal{T}}^{-1} & \cdots \\ \mathcal{A}_q P_\alpha^{-1} & 0 & \mathcal{G}\underline{\mathcal{T}}^{-1} & \cdots \\ R_0^{\frac{1}{2}}\mathcal{A}_q P_\alpha^{-1} & 0 & R_0^{\frac{1}{2}}\mathcal{G}\underline{\mathcal{T}}^{-1} & \cdots \\ \left(\begin{smallmatrix} C_\zeta & 0 \\ 0 & D_\zeta K_c \end{smallmatrix}\right) P_\alpha^{-1} & 0 & 0 & \cdots \end{pmatrix} < 0 \tag{2.33}$$

Changing the variables

One can see that the above inequality is not linear in P_α, A_c, L_c, K_c, therefore we define new LMI variables (from [10], similar to [6]).

Let us suppose, that P_α (and P_α^{-1}) is partitioned as:

$$P_\alpha = \begin{pmatrix} X & N_1 \\ N_1^\top & Z \end{pmatrix} \qquad P_\alpha^{-1} = \begin{pmatrix} Y & N_2 \\ N_2^\top & W \end{pmatrix}$$

with $X = X^\top > 0, Y = Y^\top > 0$. By substituting these into $P_\alpha \cdot P_\alpha^{-1} = I$ one can see that

$$I - X \cdot Y = N_1 \cdot N_2^\top \quad (= N_2 N_1^\top) \tag{2.34}$$

We also define the matrices

$$F_1 := \begin{pmatrix} X & I \\ N_1^\top & 0 \end{pmatrix}, \qquad F_2 := \begin{pmatrix} I & Y \\ 0 & N_2^\top \end{pmatrix}.$$

Using the condition $P_\alpha \cdot P_\alpha^{-1} = I$, the following identities can be easily derived.

$$P_\alpha^{-1}F_1 = F_2 \qquad F_1^\top P_\alpha^{-1} F_1 = \begin{pmatrix} X & I \\ I & Y \end{pmatrix} \tag{2.35}$$

By including an additional inequality

$$\begin{pmatrix} X & I \\ I & Y \end{pmatrix} > 0 \tag{2.36}$$

we ensure that the matrix $I - XY$ in (2.34) is invertible. As a consequence, N_1 and N_2 are invertible, independently of the decomposition. Furthermore, note that $P > 0$ is equivalent with (2.36), since there is a congruence transformation between them.

From now on, we refer to inequality (2.36) as *invertibility*.

Now we apply a congruence transformation on (2.33) with

$$\operatorname{diag}\{F_1, I, \dots I\}.$$

The upper-left corner transforms according to

$$F_1^\top \mathcal{A} \cdot \underbrace{P_\alpha^{-1} \cdot F_1}_{F_2} = \begin{pmatrix} X & N_1 \\ I & 0 \end{pmatrix} \cdot \begin{pmatrix} A & BK_c \\ L_c C & A_c \end{pmatrix} \cdot \begin{pmatrix} I & Y \\ 0 & N_2^\top \end{pmatrix} =$$

$$\begin{pmatrix} XA + N_1 L_c C & XAY + XBK_c N_2^\top + N_1 L_c CY + N_1 A_c N_2^\top \\ A & AY + BK_c N_2^\top \end{pmatrix}.$$

Let us introduce the notations

$$\widetilde{K} := K_c N_2^\top, \tag{2.37}$$

$$\widetilde{L} := N_1 L_c, \tag{2.38}$$

$$\widetilde{A} := XAY + XB\widetilde{K} + \widetilde{L}CY + N_1 A_c N_2^\top. \tag{2.39}$$

The blocks in the first column of (2.33) transforms similarly. The matrix $\widetilde{K}$ also appears in $\mathcal{A}_q F_2$ and in $\left(\begin{smallmatrix} C_\zeta & 0 \\ 0 & D_\zeta K_c \end{smallmatrix}\right) F_2$. After the congruence transformation the inequality is linear in $X, Y, \widetilde{A}, \widetilde{K}, \widetilde{L}$ and ϵ_i^{-1} but not in τ_i^{-1} and α^{-1}. Only the first three columns are presented, except the top-left corner.

$$\begin{pmatrix} (\cdots) & * & * & \cdots \\ \mathcal{E}^\top F_1 & -\alpha S_L & * & \cdots \\ \underline{\tau}^{-1}\mathcal{H}^\top F_1 + S_0 \mathcal{A}_q F_2 & 0 & \substack{Q_0 \underline{\tau}^{-1} + \\ S_0 \mathcal{G} \underline{\tau}^{-1} + \underline{\tau}^{-1} \mathcal{G}^\top S_0^\top} & \cdots \\ 0 & 0 & \underline{\tau}^{-1} & \cdots \\ \mathcal{A}_q F_2 & 0 & \mathcal{G}\underline{\tau}^{-1} & \cdots \\ R_0^{\frac{1}{2}} \mathcal{A}_q F_2 & 0 & R_0^{\frac{1}{2}} \mathcal{G} \underline{\tau}^{-1} & \cdots \\ \left(\begin{smallmatrix} C_\zeta & 0 \\ 0 & D_\zeta K_c \end{smallmatrix}\right) F_2 & 0 & 0 & \cdots \end{pmatrix} < 0 \tag{2.40}$$

The linearity fails because of the terms $\underline{\tau}^{-1}\mathcal{H}^\top F_1$ and α.

In our calculations (similarly to Lemma 1.3.1), we eliminated the variable τ by assigning the constant values $\tau_i = 1$ for $i = 1 \dots s$. We give a sufficient condition of (2.40) in order to eliminate α.

If $0 < \hat{\alpha} \leq \alpha$ then

$$\begin{pmatrix} (\cdots) & * & * \\ \mathcal{E}^\top F_1 & -\hat{\alpha} S_L & * \\ \vdots & & \ddots \end{pmatrix} < 0 \quad \text{implies} \quad \begin{pmatrix} (\cdots) & * & * \\ \mathcal{E}^\top F_1 & -\alpha S_L & * \\ \vdots & & \ddots \end{pmatrix} < 0 \tag{2.41}$$

since the additional negative semi-definite matrix $-(\alpha - \hat{\alpha})S_L$ does not violate negative definiteness if the whole matrix.

The overall result of the transformations can be seen on Table 2.1. The negative definiteness of this matrix will be refered to as *main LMI*.

Finally, a sufficient condition for the guaranteed cost problem is obtained as a feasibility problem. Namely, find $\widetilde{A}, \widetilde{L}, \widetilde{K}, X, Y, \epsilon^{-1}, \alpha^{-1}, \hat{\alpha}$ such that $0 < \hat{\alpha} \leq \alpha$, the *main LMI* and *invertibility* hold.

Table 2.1: The main LMI

	n_x	n_x	n_w	l_p	l_p	l_q	l_q	n_x	n_u
n_x	$A^\top X + XA + \widetilde{L}C + C^\top \widetilde{L}^\top + \delta X$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
n_x	$A + \widetilde{A}^\top + \delta I$	$AY + YA^\top + B\widetilde{K} + \widetilde{K}^\top B^\top + \delta Y$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
n_w	$E^\top X$	$E^\top$	$-\hat{\alpha} S_L$	$*$	$*$	$*$	$*$	$*$	$*$
l_p	$\underline{\mathcal{T}}^{-1} H^\top X + S_0 A_q$	$\underline{\mathcal{T}}^{-1} H^\top + S_0 \left(A_q Y + B_y \widetilde{K} \right)$	0	$Q_0 \underline{\mathcal{T}}^{-1} + S_0 \mathcal{G} \underline{\mathcal{T}}^{-1} + \underline{\mathcal{T}}^{-1} \mathcal{G}^\top S_0^\top$	$*$	$*$	$*$	$*$	$*$
l_p	0	0	0	$\underline{\mathcal{T}}^{-1}$	$-\underline{\epsilon}^{-1}$	$*$	$*$	$*$	$*$
l_q	A_q	$A_q Y + B_q \widetilde{K}$	0	$\mathcal{G} \underline{\mathcal{T}}^{-1}$	0	$-\underline{\underline{\epsilon}}^{-1}$	$*$	$*$	$*$
l_q	$\sqrt{R_0} A_q$	$\sqrt{R_0} \left(A_q Y + B_q \widetilde{K} \right)$	0	$\sqrt{R_0} \mathcal{G} \underline{\mathcal{T}}^{-1}$	0	0	$-\underline{\underline{\mathcal{T}}}^{-1}$	$*$	$*$
n_x	C_ζ	Y	0	0	0	0	0	$-\frac{1}{\alpha}$	$*$
n_u	0	$D_\zeta \widetilde{K}$	0	0	0	0	0	0	$-\frac{1}{\alpha}$

$$\text{where } A_q = \begin{pmatrix} A_{q1} \\ \vdots \\ A_{qs} \end{pmatrix}, \quad B_q = \begin{pmatrix} B_{q1} \\ \vdots \\ B_{qs} \end{pmatrix} \text{ (recall (2.18))}$$

Obtain the original variables

In order to derive the original variables P, A_c, L_c, K_c from $\widetilde{A}, \widetilde{L}, \widetilde{K}, X, Y$ one has to perform the following calculations:

from (2.34)	compute N_1, N_2 via matrix decomposition,
from (2.37)	compute K_c as $\widetilde{K} \cdot \left(N_2^\top\right)^{-1}$,
from (2.38)	compute L_c as $N_1^{-1} \cdot \widetilde{L}$,
from (2.39)	compute A_c as $N_1^{-1}\left(\widetilde{A} - XAY - XB\widetilde{K} - \widetilde{L}CY\right)\left(N_2^\top\right)^{-1}$,
$Z = -N_1^\top Y \left(N_2^\top\right)^{-1}$	$\Rightarrow P = \begin{pmatrix} X & N_1 \\ N_1^\top & Z \end{pmatrix} \cdot \dfrac{1}{\alpha}$.

By successfully solving the *main LMI* and *invertibility* one can find a Lyapunov function and a dynamic feedback control which satisfies definition 2.2.1.

2.2.2 Control constraints

In this section we derive an LMI in order to fulfill (2.8).

$$z^\top P_\alpha z \geq u^\top Q_u \underbrace{u}_{=K_c \cdot \hat{x}} \quad \forall z$$

$$\Updownarrow$$

$$P_\alpha \geq \begin{pmatrix} 0 \\ K_c^\top \end{pmatrix} Q_u \begin{pmatrix} 0 & K_c \end{pmatrix}$$

$$\Updownarrow$$

$$P_\alpha \geq \begin{pmatrix} 0 \\ K_c^\top \sqrt{Q_u} \end{pmatrix} \cdot \begin{pmatrix} 0 & \sqrt{Q_u} K_c \end{pmatrix}$$

$$\Updownarrow \text{ (Schur)}$$

$$\begin{pmatrix} P_\alpha & \begin{pmatrix} 0 \\ K_c^\top \sqrt{Q_u} \end{pmatrix} \\ \begin{pmatrix} 0 & \sqrt{Q_u} K_c \end{pmatrix} & I \end{pmatrix} \geq 0$$

Now we apply a congruence transformation with $\left(\begin{smallmatrix} P_\alpha^{-1} F_1 & 0 \\ 0 & I \end{smallmatrix}\right)$ (recall (2.35) and (2.37)) which results

$$\begin{pmatrix} X & I & 0 \\ I & Y & \widetilde{K}^\top \sqrt{Q_u} \\ 0 & \sqrt{Q_u} \widetilde{K} & I \end{pmatrix} \geq 0. \tag{2.42}$$

Requiring (2.42) reduces the control constraints to finding an upper bound for $z^\top P_\alpha z = \alpha\mathcal{V}(z)$. From now on, the relation (2.42) will be refered to as *control inequality.*

2.2.3 Subsequent time intervals

Now we consider the sampling instants t_k. If one would redefine the stabilizing feedback gains K_c^k at every time instant t_k, then stability could break down. The neighbouring time intervals have to inherit some global property.

Assume that one has solved the optimization problem at time t_k, and the solution is: $P_k, \alpha_k, \ldots$. Let us introduce the following notations: $x_k := x(t_k), y_k := y(t_k)$ and $z_k := z(t_k)$. In this section we derive some inequalities which ensure the decay of the Lyapunov function between the time intervals. Formally, we require

$$z_{k+1}^\top P_k z_{k+1} \geq z_{k+1}^\top P_{k+1} z_{k+1} \tag{2.43}$$

for the new matrix P_{k+1}. From now on, we refer to the inequality (2.43) as *ellipse invariance.*

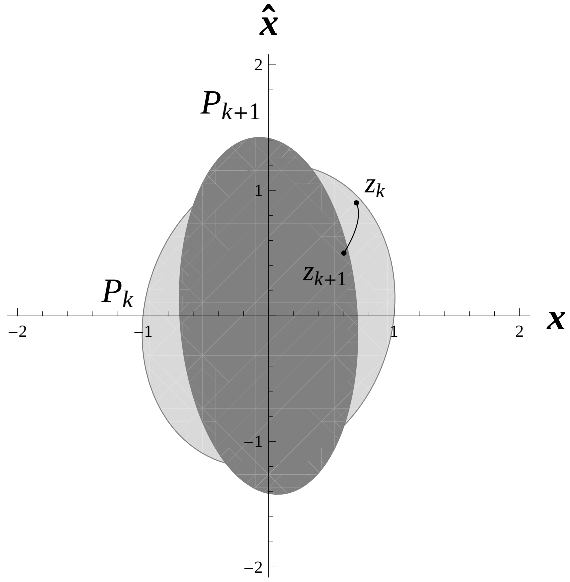


Figure 2.2: Lyapunov function fails to be continuous, but it is monotonically decreasing

In our system, the state is non accessible, therefore one cannot use z_{k+1}, but the output

$$\begin{pmatrix} y_{k+1} \\ \hat{x}_{k+1} \end{pmatrix} = \begin{pmatrix} Cx_{k+1} \\ \hat{x}_{k+1} \end{pmatrix} \in \mathbb{R}^{n_y+n_x}.$$

Clearly, the artificial observation system $\hat{x}$ is accessible, but the matrix C has usually an incomplete column rank ($n_y < n_x$). In this section we will impose a sufficient condition of *ellipse invariance* without using the state x.

We have some information about the state via the output. For a given $y \in \mathbb{R}^{n_y}$ let

$$\mathcal{X}_y := \left\{ z = \left(\begin{smallmatrix} x \\ \hat{x} \end{smallmatrix}\right) \in \mathbb{R}^{2n_x} \middle| \, Cx = y \text{ and } \hat{x} \text{ is given} \right\} \tag{2.44}$$

an affine subspace of $\mathbb{R}^{2n_x}$. This set contains all the possible states with respect to the constraint $Cx = y$ (and $\hat{x}$ known). In other words, we use as much information as we can. If the state is fully accessible, then $C = I_{n_x}$ and the set $\mathcal{X}_y$ is a single point. In general, the new ellipse should enclose a lower dimensional ellipse on an affine subspace.

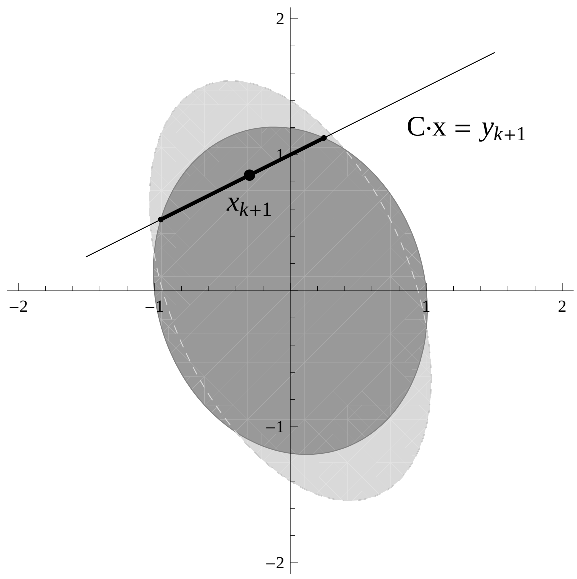


Figure 2.3: The new ellipse encloses the old ellipse on an affine object

What we will guarantee is that

$$z^\top P_{k+1} z \le z^\top P_k z \quad \forall z \in \mathcal{X}_{y_{k+1}}. \tag{2.45}$$

Let us think of the problem in a bit more abstract way. Let

$$\mathcal{C} := \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix} \quad S := P_k - P_{k+1} \quad \eta := \begin{pmatrix} y \\ \hat{x} \end{pmatrix}.$$

The set $\mathcal{X}_y$ can be parameterized as:

$$\mathcal{C}z = \eta \Leftrightarrow z = z_0 + \tilde{z}$$

where z_0 is such that $\mathcal{C}z_0 = \eta$ and $\tilde{z} \in \mathrm{Ker}(\mathcal{C})$. The vector z_0 can be computed from the given $\mathcal{C}$ (or C) and η. With these notations (2.45) is equivalent to:

$$(z_0 + \tilde{z})^\top S\,(z_0 + \tilde{z}) \geq 0 \quad \forall \tilde{z} \in \mathrm{Ker}(\mathcal{C})$$

$$\Updownarrow$$

$${z_0}^\top S z_0 + 2{z_0}^\top S\tilde{z} + \tilde{z}^\top S\tilde{z} \geq 0 \quad \forall \tilde{z} \in \mathrm{Ker}(\mathcal{C})$$

$$\Updownarrow$$

$$\beta^2 {z_0}^\top S z_0 + 2\beta {z_0}^\top S\overline{z} + \overline{z}^\top S\overline{z} \geq 0 \quad \forall \tilde{z} \in \mathrm{Ker}(\mathcal{C}) \text{ and } \beta > 0$$

where $\overline{z} = \beta \cdot \tilde{z}$. Note that $\mathcal{C}\tilde{z} = 0 \Leftrightarrow \mathcal{C}\,(\beta \cdot \tilde{z}) = 0$. The latter inequality is equivalent to

$$\begin{pmatrix} \overline{z} \\ \beta \end{pmatrix}^\top \begin{pmatrix} S & Sz_0 \\ {z_0}^\top S & {z_0}^\top S z_0 \end{pmatrix} \begin{pmatrix} \overline{z} \\ \beta \end{pmatrix} \geq 0 \quad \forall \overline{z} \in \mathrm{Ker}(\mathcal{C}) \text{ and } \beta > 0.$$

$$\Updownarrow$$

$$* \begin{pmatrix} S & Sz_0 \\ {z_0}^\top S & {z_0}^\top S z_0 \end{pmatrix} \begin{pmatrix} \mathcal{C}^\perp & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \underline{\lambda} \\ \beta \end{pmatrix} \geq 0 \quad \forall \underline{\lambda} \in \mathbb{R}^{n_x - n_y} \text{ and } \beta > 0 \tag{2.46}$$

Recall the $\mathcal{C}^\perp$ notation for the orthogonal basis of the kernel.
Note, that requiring (2.46) makes the the very same inequality hold true for $\beta \leq 0$ and requiring (2.46) for $\beta \in \mathbb{R}$, trivially implies (2.46). Therefore the condition $\beta > 0$ can be neglected and the vector $\left(\underline{\lambda}^\top, \beta\right)^\top$ runs over the $n_x - n_y + 1$ dimensional space. Consequently, an equivalent form is

$$* \begin{pmatrix} S & Sz_0 \\ {z_0}^\top S & {z_0}^\top S z_0 \end{pmatrix} \begin{pmatrix} \mathcal{C}^\perp & 0 \\ 0 & 1 \end{pmatrix} \geq 0$$

$$\Updownarrow$$

$$\begin{pmatrix} \left(\mathcal{C}^\perp\right)^\top \\ {z_0}^\top \end{pmatrix} S \overbrace{\begin{pmatrix} \mathcal{C}^\perp & z_0 \end{pmatrix}}^{D:=} \geq 0$$

$$\Updownarrow$$

$$D^\top (P_k - P_{k+1}) D \geq 0 \tag{2.47}$$

The matrix $D = \begin{pmatrix} \mathcal{C}^\perp & z_0 \end{pmatrix} \in \mathbb{R}^{n_x \times (n_x - n_y + 1)}$ can be computed from known data: $\hat{x}_{k+1}, y_{k+1}$ and C.

The condition (2.47) is equivalent to (2.45) which is weaker than $P_{k+1} \leq P_k$ on the whole $2n_x$ dimensional space, but stronger than *ellipse invariance*. The problem is, that P_{k+1} is not a decision variable in the $k+1^{\text{th}}$ LMI (recall (2.30)).

Let us multiply the latter formula by $\alpha_{k+1} > 0$ to obtain

$$D^\top (\alpha_{k+1} P_k - \alpha_{k+1} P_{k+1}) D \geq 0. \tag{2.48}$$

If $\alpha_k \leq \alpha_{k+1}$ then

$$D^\top(\alpha_k P_k - \alpha_{k+1} P_{k+1})D \geq 0$$

implies (2.48). By Schur complement the latter is equivalent to

$$\begin{pmatrix} \alpha_k D^\top P_k D & D^\top \\ D & (\alpha_{k+1} P_{k+1})^{-1} \end{pmatrix} \geq 0.$$

Now we apply a congruence transformation with F_1 (recall formula (2.35)) and we also append the condition $\alpha_k \leq \alpha_{k+1}$

$$\begin{pmatrix} \alpha_k D^\top P_k D & D^\top F_1 & 0 \\ F_1{}^\top D & \begin{pmatrix} X & I \\ I & Y \end{pmatrix} & 0 \\ 0 & 0 & \alpha_k^{-1} - \alpha_{k+1}^{-1} \end{pmatrix} \geq 0 \tag{2.49}$$

Note that X, Y and α_{k+1}^{-1} are truly the variables of the *main LMI* and α_k, P_k, D are known from the previous time segment. Also the matrix N_1 becomes a decision variable (N_1 appears in F_1). Hence, N_2 is computed as $N_2^\top = N_1^{-1}(I - XY)$, when we obtain the original variables.

The inequality $N_1 + N_1^\top > 0$ guarantees $\det N_1 \neq 0$, however we used the simplification that $N_1 = n_1 \cdot I$ with $n_1 > 0$ for numerical reasons. Even if the matrix N_1 only has one degree of freedom the feasibility of (2.49) does not break, and it has a good effect on the condition numbers of the matrices A_c, K_c, L_c.

From now on, we refer inequality (2.49) as *continuity condition.*
Note that $\hat{\alpha} \leq \alpha_0$ was required on the first time step and $\alpha_k \leq \alpha_{k+1}$ in the further steps. Therefore, $\hat{\alpha} \leq \alpha_k$ is maintained in the whole process.

2.3 Statement of result

In this section we assemble the matrices, what we found with the former inequalities, into an applicable method and prove some convergence properties.

Theorem 2.3.1. *Suppose that the* main LMI *and* invertibility *hold true for system (2.1)-(2.4) with uncertainty constrains (2.5) where Ω satisfies Assumption 2.Let us apply the un-updated version of the controller (2.12)-(2.13) on $[0, \infty)$. Then the cost function (2.11) is bounded for any admissible initial state, disturbance and uncertainty. If the disturbances are in Class Δ_{II} (see definition 2.1.2), then the upper bound is*

$$z(0)^\top P z(0) + \eta.$$

If the disturbances are in Class Δ_I (see definition 2.1.1), then the upper bound is $z(0)^\top Pz(0)$ and the origin of the closed-loop uncertain system is asymptotically stable and the ellipsoid

$$\left\{z \in \mathbb{R}^{2n_x} \,\middle|\, z^\top Pz \le z(0)^\top Pz(0)\right\}$$

is within the basin of attraction.

Proof. The *main LMI* gives us the Lyapunov function $\mathcal{V}(z) = z^\top Pz$ and the control $k(z) = K_c \cdot \hat{x}$ which satisfy (2.23) (recall the definition 2.2.1). This means that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}(z(t)) + z(t)^\top \underbrace{\begin{pmatrix} Q_L & 0 \\ 0 & {K_c}^\top R_L K_c \end{pmatrix}}_{\overline{Q}} z(t) - w(t)^\top S_L w(t) \le -\delta\mathcal{V}(z(t)) \tag{2.50}$$

By neglecting the right-hand side and integrating from 0 to $T > 0$ it follows that

$$\mathcal{V}(z(T)) - \mathcal{V}(z(0)) + \int_0^T \|z(t)\|^2_{\overline{Q}}\,\mathrm{d}t - \int_0^T \|w(t)\|^2_{S_L}\mathrm{d}t \le 0. \tag{2.51}$$

Omitting the first nonnegative term on the left-hand side, we obtain that

$$\int_0^T \mathcal{L}(z(t), w(t))\mathrm{d}t \le \mathcal{V}(z(0)) \tag{2.52}$$

or alternatively

$$\int_0^T \|z(t)\|^2_{\overline{Q}}\,\mathrm{d}t \le \mathcal{V}(z(0)) + \int_0^T \|w(t)\|^2_{S_L}\mathrm{d}t \tag{2.53}$$

for all $T > 0$. If the disturbance is in Class Δ_{II} then the integral on the right-hand side converges as $T \to \infty$. Therefore the left-hand side is also bounded, consequently it is convergent. Therefore the cost function is bounded by

$$\mathcal{V}(z(0)) + \eta$$

which proves the statement for the Class Δ_{II} case.

On the other hand one can rearrange (2.50) using the definition of the cost function (2.11).

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}(z(t)) + L(x(t), K_c\hat{x}(t), w(t)) \le -\delta\mathcal{V}(z(t))$$

If the disturbances are in Class Δ_I (see definition 2.1.1), then $L(x(t), K_c\hat{x}(t), w(t)) \ge 0$ and we can neglect it. This gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}(z(t)) \le -\delta\mathcal{V}(z(t)). \tag{2.54}$$

We have required $P > 0$, therefore the function $\mathcal{V}$ is positive definite, and the Lyapunov theorem can be applied. Hence, the origin is asymptotically stable with a basin of attraction containing the ellipse

$$\left\{ z \in \mathbb{R}^{2n_x} \middle| z^\top P z \leq z(0)^\top P z(0) \right\}.$$

From (2.52) one can see that the cost functional is bounded. From the Class Δ_I condition we know that the running cost is non-negative, thus the total cost

$$\int_0^\infty \mathcal{L}(z(t), w(t))\mathrm{d}t$$

is convergent and bounded by $\mathcal{V}(z(0))$. □

Statement 2.3.2. *If the initial state $x(0) = x_0$ is known, then the additional LMI*

$$\begin{pmatrix} 1 & * & * \\ Xx_0 & X & I \\ x_0 & I & Y \end{pmatrix} \geq 0 \tag{2.55}$$

ensures the following initial bound for the Lyapunov function.

$$\alpha_0 \mathcal{V}(z(0)) \leq 1$$

Proof. We shall rewrite the inequality $\alpha_0 \mathcal{V}(z_0) = z_0^\top P_\alpha z_0 \leq 1$ in an equivalent form, similarly to *continuity condition*.

$$1 \geq z_0^\top P_\alpha z_0 \Leftrightarrow \begin{pmatrix} 1 & z_0^\top \\ z_0 & {P_\alpha}^{-1} \end{pmatrix} \geq 0$$

$$\Updownarrow$$

$$\begin{pmatrix} 1 & z_0^\top F_1 \\ {F_1}^\top z_0 & \begin{pmatrix} X & I \\ I & Y \end{pmatrix} \end{pmatrix} \geq 0.$$

By substituting $z_0 = \left({x_0}^\top, 0 \right)^\top$ into $F_1^\top z_0$ one can get (2.55). □

Corollary 2.3.3. *If the disturbance is in Class Δ_I then the* control inequality*, in addition to the formers, guarantees the control constraint (2.8).*
If the disturbance is in Class Δ_{II} then the control inequality *implies a weaker bound*

$$\|u(t)\|^2_{Q_u} \leq 1 + \eta.$$

Proof. The *control inequality* guarantees that

$$\|u(t)\|^2_{Q_u} \leq z(t)^\top P_\alpha z(t) \quad \forall t \geq 0.$$

For Class Δ_I:

$$z(t)^\top P_\alpha z(t) \overset{\text{Theorem 2.3.1}}{\leq} z(0)^\top P_\alpha z(0) \overset{\text{Statement 2.3.2}}{\leq} 1$$

which concludes (2.8).
The Class Δ_{II} condition and (2.51) concludes

$$z(t)^\top P_\alpha z(t) \leq z(0)^\top P_\alpha z(0) + \eta \overset{\text{Statement 2.3.2}}{\leq} 1 + \eta.$$

Thus the weaker bound is also proved. $\square$

Now the time intervals will be concerned, recall the time dependent controller (2.12)-(2.13). We can observe that the matrices K_c and P may vary in time, but the *main LMI* does not. Therefore we need a suitable version of a Lyapunov theorem.

The original version of the following Lemma has been published in [5]. It has been fitted to the present situation. Consider the system described by

$$\left.\begin{aligned} \dot{z} &= \mathcal{A}_k z + \mathcal{E} w + \mathcal{H} p \\ q &= \mathcal{A}_{qk} z + \mathcal{G} p \end{aligned}\right\} \quad t \in [t_k, t_{k+1}), \tag{2.56}$$

where $\mathcal{A}_k$ and $\mathcal{A}_{qk}$ are constant matrices, $\left(p^\top, q^\top\right)^\top \in \Omega$, and Ω is defined by (2.5) satisfying Assumption 2. Let

$$\mathcal{V}(t, z) = z^\top P_k z^\top, \quad \text{if } t \in [t_k, t_{k+1}).$$

Lemma 2.3.4. *Suppose that there exist positive numbers $\gamma_1, \gamma_2, \delta$ such that*

$$\gamma_1 I \leq P_k \leq \gamma_2 I \qquad \forall t \in \mathbb{N}. \tag{2.57}$$

Let the function $\overline{V} : t \mapsto \mathcal{V}(t, z(t))$ be continuous from the right for $z(\cdot)$ satisfying (2.56), absolutely continuous for $t \neq t_k$ and satisfies

$$\lim_{t \to t_k^-} \overline{V}(t) \geq \overline{V}(t_k). \tag{2.58}$$

(i) If along (2.56)

$$\dot{\overline{V}}(t) + \delta \overline{V}(t) \leq 0 \qquad \textit{for almost all } t \tag{2.59}$$

then $\overline{V}(t) \leq e^{-\delta t} \overline{V}(0)$ i.e. $\|z(t)\|^2 \leq \frac{\gamma_2}{\gamma_1} e^{-\delta t} \|z(0)\|^2$.

(ii) If along (2.56)

$$\dot{\overline{V}}(t) + \mathcal{L}(z(t), w(t)) \leq -\delta \overline{V}(t) \qquad \textit{for almost all } t \tag{2.60}$$

then

$$\limsup_{t \to \infty} \int_0^t \mathcal{L}(z(\tau), w(\tau)) \mathrm{d}\tau \leq \overline{V}(0)$$

and the trajectory of (2.56) remains bounded on $[0, \infty)$.

Proof.

(i) From (2.57) and (2.59) it follows that

$$\gamma_1 \|z(t)\|^2 \leq \overline{V}(t) \leq e^{-\delta(t-t_k)} \overline{V}(t_k) \qquad\qquad t \in [t_k, t_{k+1})$$

By taking into account (2.58), it follows that

$$e^{-\delta(t-t_k)} \overline{V}(t_k) \leq e^{-\delta(t-t_{k-1})} \overline{V}(t_{k-1}) \leq \ldots \leq e^{-\delta t} \overline{V}(0) \leq \gamma_2 e^{\delta t} \|z(0)\|^2$$

and $\|z(t)\|^2 \leq \frac{\gamma_2}{\gamma_1} e^{-\delta t} \|z(0)\|$.

(ii) Let $N \gg 1$ be given, and let us integrate (2.60) from 0 to t where $t \in [t_{N-1}, t_N)$. Then we obtain

$$\overline{V}(t) - \overline{V}(t_{N-1}) + \overline{V}\left(t_{N-1}^-\right) \mp \ldots - \overline{V}(0) + \int\limits_0^t \mathcal{L}(z(\tau), w(\tau)) \mathrm{d}\tau \leq 0. \quad (2.61)$$

Since $\overline{V}(t) \geq 0$ and (2.58) is supposed to be valid, we can conclude that

$$\limsup_{t \to \infty} \int\limits_0^t \mathcal{L}(z(\tau), w(\tau)) \mathrm{d}\tau \leq \overline{V}(0),$$

which means that $\overline{V}(0)$ is an upper bound of the cost functional. On the other hand, (2.61) involves that

$$\overline{V}(t) \leq \overline{V}(t) + \int\limits_0^t \|z(\tau)\|_Q^2 \mathrm{d}\tau \leq \overline{V}(0) + \int\limits_0^t \|w(\tau)\|_{S_L}^2 \mathrm{d}\tau$$

therefore, for $w \in \Delta_{II}$, we get that

$$\gamma_1 \|z(t)\|^2 \leq \overline{V}(0) + \eta, \qquad\qquad \forall t \geq 0$$

thus the trajectory of (2.56) remains bounded.

□

Remark 2.3.5. *If only the case* (ii) *is valid then function* $\overline{V}(\cdot)$ *is no longer monotonically decreasing within* $[t_k, t_{k+1})$.

Remark 2.3.6. *The* ellipse invariance *is exactly (2.58) for which* continuity condition *is a sufficient condition.*

Lemma 2.3.4 is clearly applicable to our system and satisfies the control task. Let us summarize the overall results in the following theorem.

Theorem 2.3.7. *Consider the system (2.1)-(2.4) where the uncertainties fulfill Assumption 2. If we apply the control policy (2.12)-(2.13) with requiring inequality (2.55) and* $\hat{\alpha} \leq \alpha_0$ *the at the first time instant,* continuity condition *for the subsequent sampling instants, furthermore* main LMI *and* invertibility *for every sampling instant then on can state the followings.*

- *For* $w \in \Delta_I$ *the origin of the closed-loop uncertain system is asymptotically stable, the ellipsoid*

$$\left\{ z \in \mathbb{R}^{2n_x} \middle| z^\top P z \leq z(0)^\top P z(0) \right\}$$

  is within the basin of attraction, and the cost functional (2.11) is bounded by $\frac{1}{\alpha_k}$.

- *If* $w \in \Delta_{II}$ *then the trajectories remain bounded and the cost functional (2.11) is bounded by* $\frac{1}{\alpha_0} + \eta$.

Proof. See Lemma 2.3.4. □

Remark 2.3.8. *The* control inequality *in addition to the former ones also guarantees the statement of Corollary 2.3.3.*

Remark 2.3.9. *The* main LMI, invertibility *and* continuity condition *are feasible at the sampling instant* t_{k+1} *if it was feasible at* t_k*, since the old solution (with* $P_k, \alpha_k \ldots$*) satisfy the new inequalities also.*

Chapter 3

Applications, simulations

In this section we apply the proposed method to control certain systems. We cite the original example and the way we could improve them, or numerical problems, where occured.

We will use our notations (see (2.1)-(2.4)) not the original notations of the cited papers.
The simulations follow the following steps:

1. System setup, $k = 0$
   - Assign the constant (given) matrices of the system.
   - Assign the uncertainty and disturbance functions.
   - Assign the initial condition x_0.
2. Solve the *main LMI*, *invertibility*, (2.55) and *control inequality*
3. Simulate the augmented closed loop system for $t \in [t_k, t_{k+1})$ and increase k.
4. Solve the *main LMI*, *invertibility* and *continuity condition* using the previous data, and *control inequality*
5. Go to step 3.

Note, that the purpose of the uniform decay rate δ is only theoretical. In the following calculations we always choose $\delta = 10^{-4}$.

3.1 Flexible link robotic arm [3]

The system is a Lure system.

$$\dot{x} = Ax + Bu + Ew + Hp \tag{3.1}$$
$$y = Cx$$
$$q = A_q x$$
$$\zeta = \begin{pmatrix} C_\zeta x \\ D_\zeta u \end{pmatrix} \tag{3.2}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.25 & 48.6 & 0 \\ 0 & 0 & 0 & 1 \\ 19.5 & 0 & -16.7 & 0 \end{pmatrix},\ B = \begin{pmatrix} 0 \\ 21.6 \\ 0 \\ 0 \end{pmatrix},\ E = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},\ H = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -3.33 \end{pmatrix}$$
$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},\ A_q = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}$$

and the cost matrices are chosen to be

$$Q_L = C_\zeta^\top C_\zeta = \begin{pmatrix} 1 & & & \\ & 0.1 & & \\ & & 1 & \\ & & & 0.1 \end{pmatrix},\ R_L = D_\zeta^\top D_\zeta = 0.1,\ S_L = 1.$$

The uncertainty is a nonlinearity on the right-hand side:

$$p(x) = \sin x_3 + x_3, \quad q = A_q x = x_3 \tag{3.3}$$

In paper [3], it has been assumed that the whole state is available for feedback, i.e. $u = K \cdot x$ can be used, which is a significantly simpler case. In this section one can see simulations for constant zero, Class Δ_I and Class Δ_{II} disturbances. In [3] no disturbance has been allowed.

The uncertainty (3.3) is sector bounded, since

$$\begin{pmatrix} p \\ q \end{pmatrix}^\top \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} \leq 0,$$

Thus it belongs to the class investigated in the previous section. The set Ω is given by $Q_0 = -1, S_0 = 1, R_0 = 0$ (and $s = 1$). We assign $G = 0$, because the uncertain input does not appear in q, therefore Assumption 2 trivially holds true.

We require the control constraint $-1.5 \leq u \leq 1.5$ by assigning $Q_u = 1.5^{-2}$. The initial state is $x_0 = (1.2, 0, 0, 0)^\top$ and the system was simulated for 6 seconds, by dividing this interval into 10 equal time intervals.

Without disturbance

In the first simulation the disturbance is $w(t) \equiv 0$.

The lower bound $\hat{\alpha}$ was initially assigned to 0.01. The result of the calculations verified that indeed $\hat{\alpha} \leq \alpha_0$.

The results are shown on page 49. The black graphs are the NMPC, the grays are the un-updated control. The gray and black graphs overlap on the first time interval, but the update of the controller separates them later.

Disturbance of Class Δ_I

For simulating a Class Δ_I disturbance, we assigned $E = (0.1, 0, 0, 0)^\top$ and

$$w(t, x(t)) = 0.05 \cdot \|x(t)\|$$

which clearly satisfies (2.9) with $\gamma_\Delta = 0.5$ (recall C_ζ).

The lower bound $\hat{\alpha}$ was assigned to 0.001, smaller then in the previous simulation. The decreasing of $\hat{alpha}$ was necessary to obtain a feasible LMI.

The results are shown on page 50. One can see that the Lyapunov function on Figure 3.2(d) decays monotonically, however not so sharply as on Figure 3.1(d).

Disturbance of Class Δ_{II}

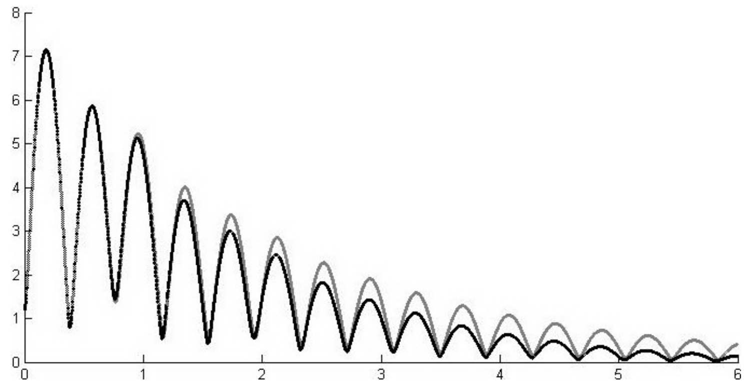
Let us re define the disturbance as

$$w(t) := \begin{cases} \frac{100}{t+1} & \text{if } 0.1 \leq \{t^{1.5}\} \leq 0.3 \\ 0 & \text{otherwise} \end{cases} \tag{3.4}$$

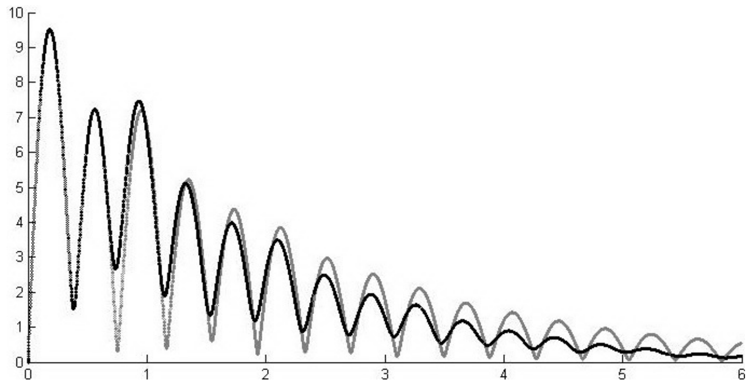
without changing any of the other matrices and constants. The bracket $\{\cdot\}$ denotes the fraction part. It can be checked that this predefined function is in Class Δ_{II}.

The results are shown on page 52. One can see that the Lyapunov function on Figure 3.4(d) is not monotonically decreasing, when the disturbance is active.

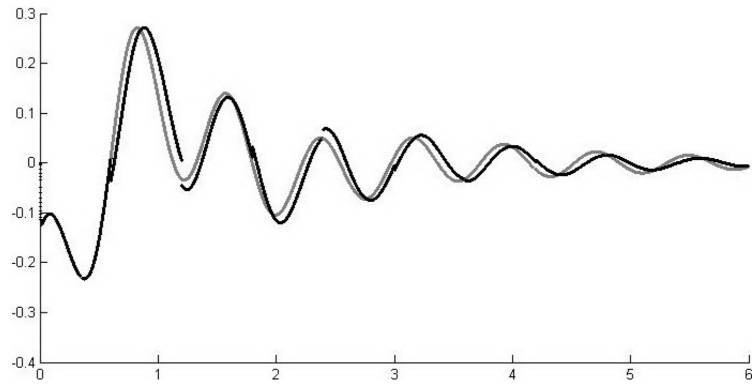
Figure 3.1: Simulation of system 3.1, $w \equiv 0$



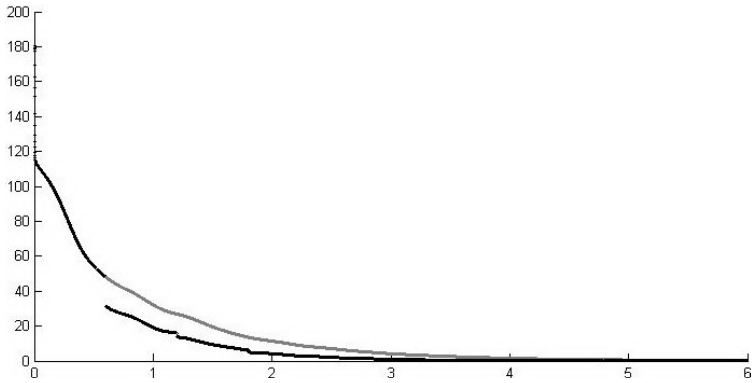
(a) $\|x(t)\|$



(b) $\|\hat{x}(t)\|$

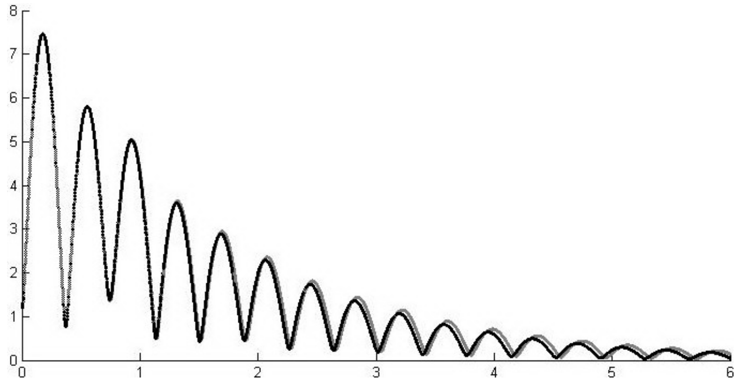


(c) $u(t) = K_c^k \cdot \hat{x}(t)$

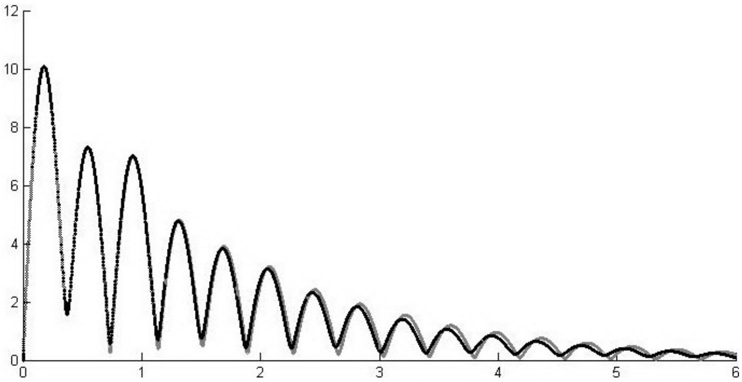


(d) $\overline{V}(t) = z(t)^\top P_k z(t)$

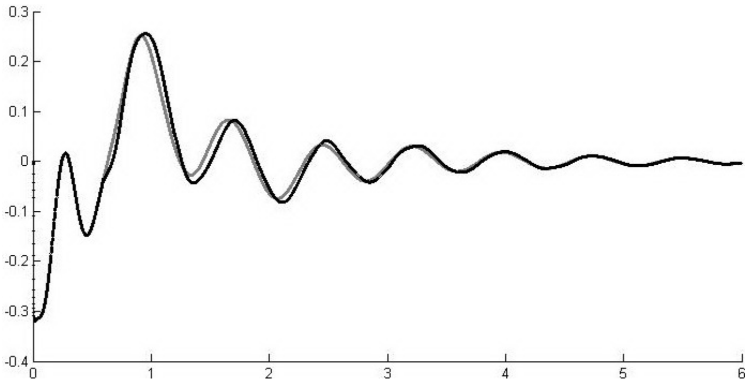
Figure 3.2: Simulation of system 3.1, $w \in \Delta_I$



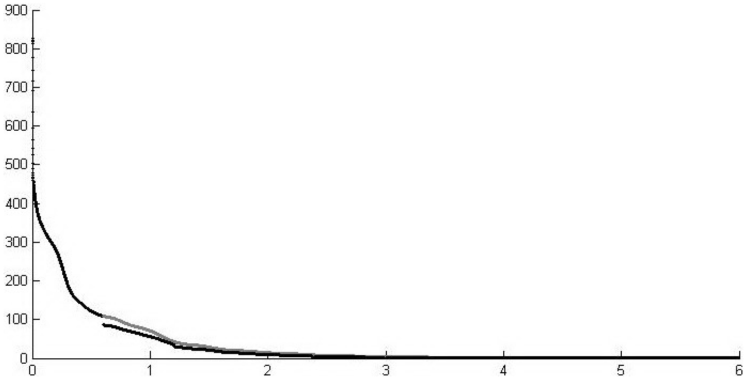
(a) $\|x(t)\|$



(b) $\|\hat{x}(t)\|$



(c) $u(t) = K_c^k \cdot \hat{x}(t)$



(d) $\overline{V}(t) = z(t)^\top P_k z(t)$

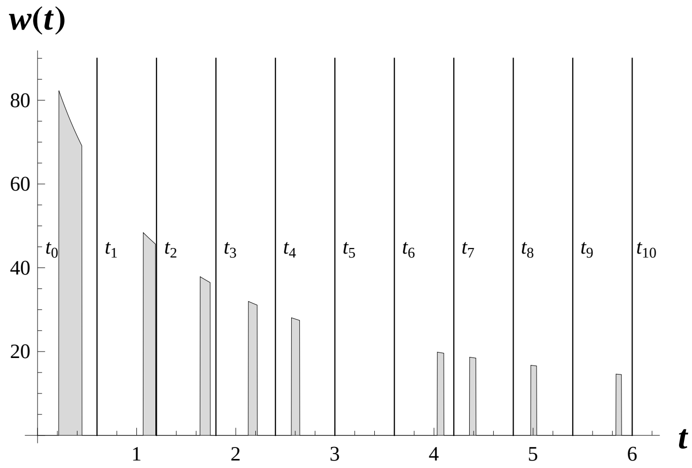


Figure 3.3: Disturbance function $w \in \Delta_{II}$

3.2 Stabilization of Dynamically Positioned Ships

The below presented system is from [11], refered from [4]. The dynamics of the 6 dimensional system is described by the following, nonlinear ODE, with 3 dimensional control u:

$$\begin{pmatrix} \dot{n} \\ \dot{e} \\ \dot{\psi} \\ \dot{\mu} \\ \dot{v} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \cos\psi & -\sin\psi & 0 \\ 0 & 0 & 0 & \sin\psi & \cos\psi & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -0.0358 & 0 & 0 & -0.0797 & 0 & 0 \\ 0 & -0.0208 & 0 & 0 & -0.0818 & -0.1224 \\ 0 & -0.0394 & 0 & 0 & -0.2254 & -0.2468 \end{pmatrix} \cdot \begin{pmatrix} n \\ e \\ \psi \\ \mu \\ v \\ r \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.9215 & 0 & 0 \\ 0 & 0.7802 & 1.4811 \\ 0 & 1.4811 & 7.4562 \end{pmatrix} \cdot u$$

The control task is to stabilize the constant zero solution. Unlike in [11], the state is known up to the output

$$y = \begin{pmatrix} n \\ e \\ \psi \end{pmatrix} = \underbrace{\begin{pmatrix} I & 0 \end{pmatrix}}_{C:=} \cdot \begin{pmatrix} n \\ e \\ \psi \\ \mu \\ v \\ r \end{pmatrix}. \tag{3.5}$$

We want to separate the nonlinear effects, like in (1.15), therefore we define the uncertain output q and the uncertainty p as:

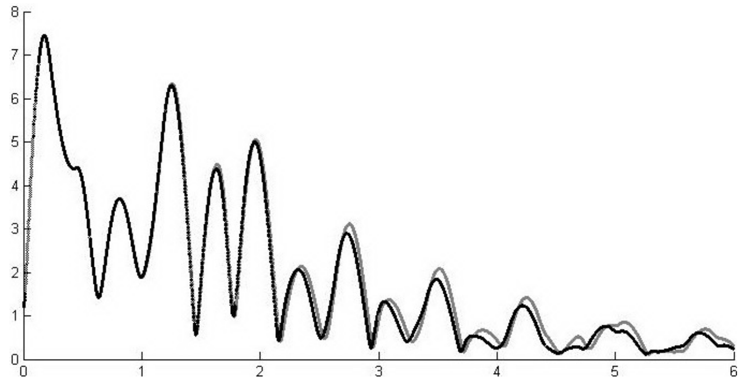
$$q := \begin{pmatrix} \mu \\ v \end{pmatrix} \tag{3.6}$$

$$p := \begin{pmatrix} (\cos\psi - 1) \cdot \mu - \sin\psi \cdot v \\ \sin\psi \cdot \mu + (\cos\psi - 1) \cdot v \end{pmatrix} \tag{3.7}$$

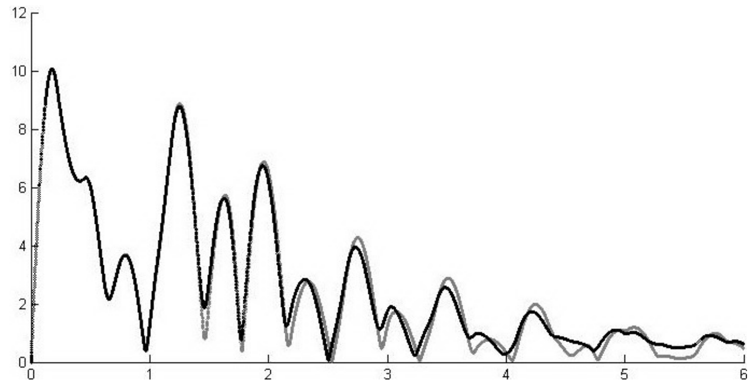
The uncertain dynamics reads as:

$$\begin{aligned} \dot{x} &= Ax + Bu + Hp \\ y &= Cx \\ q &= A_q x \end{aligned}$$

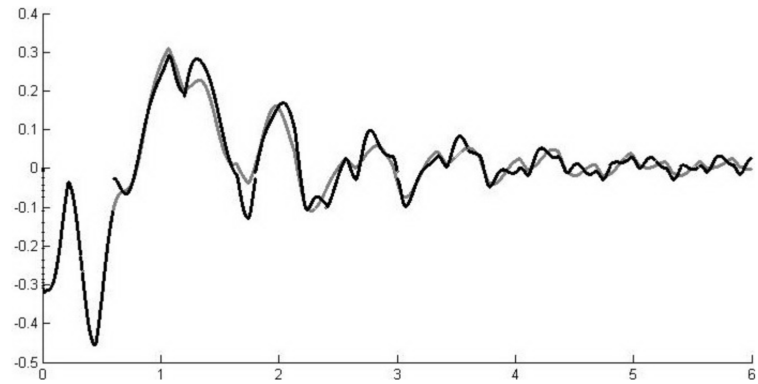
Figure 3.4: Simulation of system 3.1, $w \in \Delta_{II}$



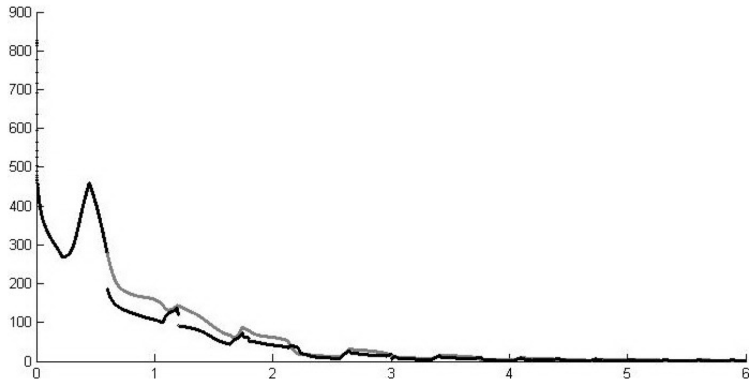
(a) $\|x(t)\|$



(b) $\|\hat{x}(t)\|$



(c) $u(t) = K_c^k \cdot \hat{x}(t)$



(d) $\overline{V}(t) = z(t)^\top P_k z(t)$

where the state is $x = (n, e, \psi, \mu, v, r)^\top$ and

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -0.0358 & 0 & 0 & -0.0797 & 0 & 0 \\ 0 & -0.0208 & 0 & 0 & -0.0818 & -0.1224 \\ 0 & -0.0394 & 0 & 0 & -0.2254 & -0.2468 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.9215 & 0 & 0 \\ 0 & 0.7802 & 1.4811 \\ 0 & 1.4811 & 7.4562 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad A_q = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

We assign the cost matrices as:

$$C_\zeta = I_6 \qquad\qquad D_\zeta = I_3.$$

We require $\|u\|^2 \leq 10$ via $Q_u := 0.1 \cdot I_3$. One can check that the uncertainty fulfills $\|p\|^2 \leq 4\|q\|^2$, therefore we let:

$$Q_0 = -I, \quad R_0 = 4 \cdot I, \quad S_0 = 0$$

in the definition of the uncertainty set Ω. In other words, the uncertainty is norm bounded. We do not apply any disturbance, hence $n_w = 0$.

The corresponding LMI, defined by the above matrices, was found infeasible, but we shall describe an other approach to control the system.

Let us consider the uncertainty p as an exogenous disturbance w. Namely, let $l_p = l_q = 0, n_w = 2$ and

$$E := \begin{pmatrix} 0.01 & 0 \\ 0 & 0.01 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since the nonlinear function $p(x)$ is norm bounded, the corresponding disturbance $w(t) := 100 \cdot p(x(t))$ is in Class Δ_I with $S_L := \frac{0.01}{4} \cdot I_2$, recall (2.9):

$$\|w\|^2_{S_L} = \frac{0.01}{4}\|w\|^2 = \frac{1}{4}\|p(x)\|^2 \leq \underbrace{\mu^2 + v^2}_{\|q\|^2} \leq \|x\|^2 + \|u\|^2 = \|\zeta\|^2.$$

The matrices A, B, C_ζ, D_ζ and Q_u remain the same as before, but A_q, G, H, R_0, S_0 and Q_0 become obsolete.

With this trick, we managed to control the system, but we are unable to apply a Class Δ_{II} disturbance.

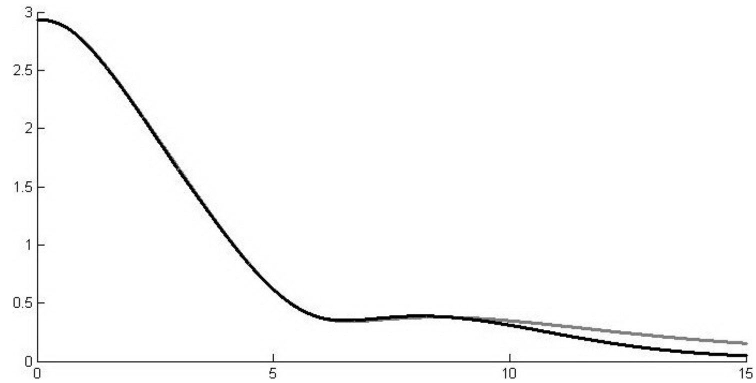
The initial state is $x_0 = (-2, 2, -\frac{\pi}{4}, 0, 0, 0)^\top$ and we simulated the system for 15 seconds, by dividing this interval into 20 equal time instances. The auxiliary variable $\hat{\alpha}$ was set to 10^{-3}.

One can see the results on page 55. The black graphs are the NMPC, the gray ones are the un-updated control. Since a Class Δ_I disturbance is applied, the Lyapunov function is monotonically decreasing with jumps.

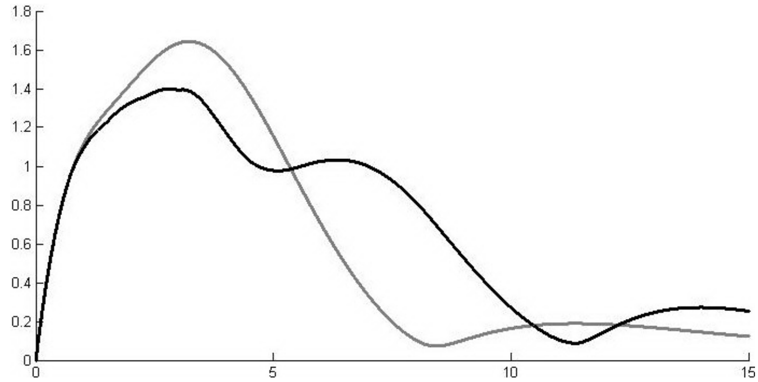
At 3 out of 20 times the LMI solver[1] reported numeric instability and the update of the controller failed. At these time instances the old controller was used from the previous time interval. When the numerical stability regained, the update was continued.

[1]SeDuMi 1.3, online available at `http://sedumi.ie.lehigh.edu/`

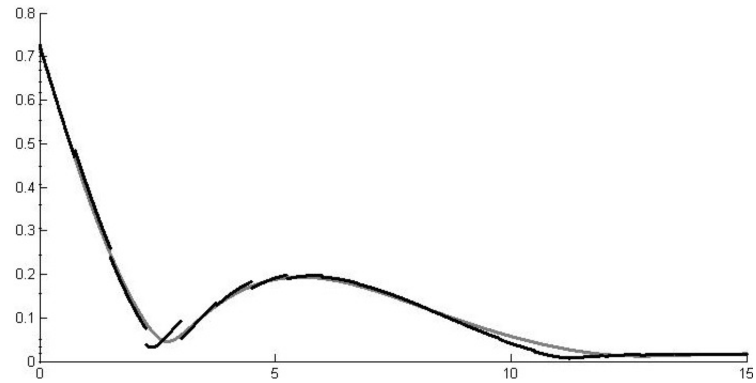
Figure 3.5: Simulation of system 3.2



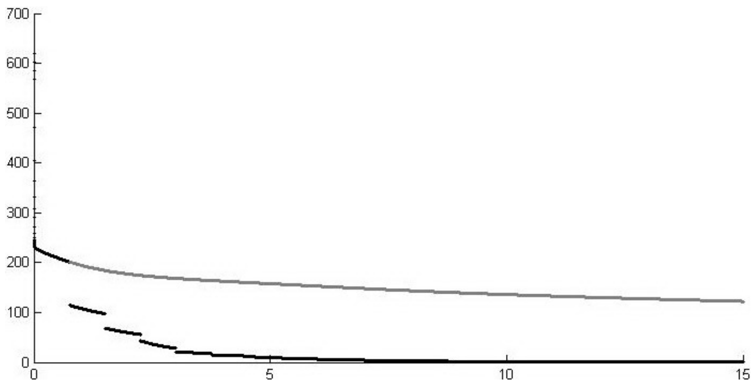
(a) $\|x(t)\|$



(b) $\|\hat{x}(t)\|$



(c) $\|u(t)\| = \left\|K_c^k\right\| \cdot \hat{x}(t)$



(d) $z(t)^\top P_k z(t)$

3.3 Double inverted pendulum

Consider a double pendulum with two masses, in 2 dimensions, hanging on two limbs and moving freely under the force of gravity. The system has two degrees of freedom ϕ_1 and ϕ_2, these the angles between the limbs and the vertical direction, respectively. The limbs have zero mass and every resistance is neglected. There control is torque applied in the lower pivot point. The $\phi_1 = \phi_2 = 0$ is an unstable

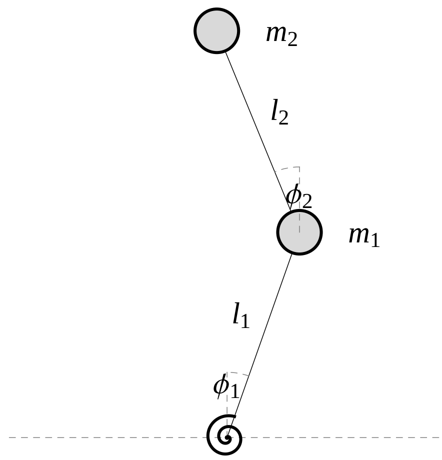


Figure 3.6: Double pendulum

equilibrium point of the system, what we will stabilize.

We will derive the equation of motion via Hamiltonian dynamics, then consider the nonlinearity as an uncertainty.

The configuration space is a torus $\phi = \begin{pmatrix} \phi_1 \\ \phi_1 \end{pmatrix} \in \mathbb{T}^2$. The Cartesian coordinates of the masses ($\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^2$) can be expressed with the generalized coordinates:

$$\mathbf{x}_1(\phi_1, \phi_2) = \begin{pmatrix} x_1^1 \\ x_1^2 \end{pmatrix} = l_1 \begin{pmatrix} \sin\phi_1 \\ \cos\phi_1 \end{pmatrix}, \quad \mathbf{x}_2(\phi_1, \phi_2) = \begin{pmatrix} x_2^1 \\ x_2^2 \end{pmatrix} = \mathbf{x}_1(\phi_1, \phi_2) + l_2 \begin{pmatrix} \sin\phi_2 \\ \cos\phi_2 \end{pmatrix}$$

The Lagrangian function can be obtained:

$$\begin{aligned} T(\phi_1, \phi_2, \dot{\phi}_1, \dot{\phi}_2, t) &= \frac{1}{2} m_1 \left\| \nabla \mathbf{x}_1(\phi_1, \phi_2) \cdot \dot{\phi} \right\|^2 + \frac{1}{2} m_2 \left\| \nabla \mathbf{x}_2(\phi_1, \phi_2) \cdot \dot{\phi} \right\|^2, \\ V(\phi_1, \phi_2, \dot{\phi}_1, \dot{\phi}_2, t) &= m_1 \cdot g \cdot x_1^2(\phi_1, \phi_2) + m_2 \cdot g \cdot x_2^2(\phi_1, \phi_2) - \underbrace{\phi_1 u(t)}, \\ L(\phi_1, \phi_2, \dot{\phi}_1, \dot{\phi}_2, t) &= T(\phi_1, \phi_2, \dot{\phi}_1, \dot{\phi}_2, t) - V(\phi_1, \phi_2, \dot{\phi}_1, \dot{\phi}_2, t). \end{aligned}$$

Note that the controlling torque is manifested as an explicitly time-dependent term in the potential function V.

The Hamiltonian function can be derived via Legendre transform[2]:

$$\begin{aligned}H(\phi_1, p_1, \phi_2, p_2, t) =& gl_1(m_1+m_2)\cos\phi_1 - \\ &+ \frac{l_1^2 p_2^2(m_1+m_2) - 2l_1 l_2 m_2 p_1 p_2 \cos(\phi_1-\phi_2) + l_2^2 m_2 p_1^2}{l_1^2 l_2^2 m_2(2m_1+m_2-m_2\cos(2\phi_1-2\phi_2))} + \\ &+ gl_2 m_2 \cos\phi_2 - \phi_1 u(t)\end{aligned}$$

Note that $m_1 > 0$ and $m_2 > 0$ ensure that the denominator is uniformly bounded away from 0 and consequently the Hamiltonian is smooth. Let us denote the coordinates of the phase space with

$$\Phi := \begin{pmatrix} \phi_1 \\ p_1 \\ \phi_2 \\ p_2 \end{pmatrix}. \tag{3.8}$$

Using the Hamiltonian function, the dynamics of the system is described by the following ODE:

$$\dot{\Phi}(t) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial}{\partial\phi_1} H(\Phi(t),t) \\ \frac{\partial}{\partial p_1} H(\Phi(t),t) \\ \frac{\partial}{\partial\phi_2} H(\Phi(t),t) \\ \frac{\partial}{\partial p_2} H(\Phi(t),t) \end{pmatrix}$$

The parameters of the simulated system were the followings:

$$m_1 = 1, \quad l_1 = 1, \quad m_2 = 1, \quad l_2 = 1, \quad g = 9.81$$

The vector-valued function on the right-hand side is strongly nonlinear in the generalized coordinates and momenta, however linear in the control $u(t)$.

$$\dot{\Phi}(t) = \underline{F}\left(\Phi(t)\right) + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} u(t)$$

We separate the linear term from the right-hand side, like in (1.15).

$$\underline{F}(\Phi) = \mathrm{D}_0\, \underline{F} \cdot \Phi + \overbrace{\underline{F}(\Phi) - \mathrm{D}_0\, \underline{F} \cdot \Phi}^{\gamma(\Phi):=} \tag{3.9}$$

The γ function is the uncertainty in the system. We cannot obtain exact estimates on γ but we know that

- γ is smooth,
- $\gamma(0) = 0$ and
- $\mathrm{D}_0\, \gamma = 0$.

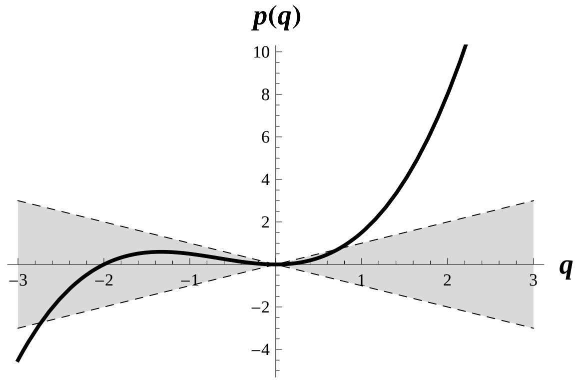


Figure 3.7: Non-conic nonlinearity

Due to the quadratic terms in γ, we failed to enclose the nonlinearity into a cone. However one can state, that for every $r > 0$, there exist a small neighbourhood $0 \in \mathcal{N}_r \subseteq \mathbb{R}^4$ such that following holds true for all $\Phi \in \mathcal{N}_r$.

$$\|\gamma(\Phi)\|^2 \leq r \cdot \|\Phi\|^2 \tag{3.10}$$

which is equivalent to

$$\begin{pmatrix} \gamma(\Phi) \\ \Phi \end{pmatrix}^\top \begin{pmatrix} -I & 0 \\ 0 & r \cdot I \end{pmatrix} \begin{pmatrix} \gamma(\Phi) \\ \Phi \end{pmatrix} \geq 0. \tag{3.11}$$

One shall choose r as big as possible, to enlarge the set $\mathcal{N}_r$. As r grows we have less and less information about the uncertainty, therefore the feasibility breaks down.

Like in Section 3.2, the disturbance based approach is used, to treat the nonlinearity. The simulated system is the following.

$$\begin{aligned} \dot{\Phi} &= A\Phi + Bu + Ew(\Phi) \\ y &= C\Phi \end{aligned}$$

where $w(\Phi) = 1000 \cdot \gamma(\Phi)$ and

$$A = \begin{pmatrix} 0 & 1 & 0 & -1 \\ 2g & 0 & 0 & 0 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & g & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \; E = 0.001 \cdot I_4, \; C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

[2]The calculations for the Hamiltonian system was done with Wolfram Mathematica.

Formally there is no uncertainty applied, therefore $l_p = l_q = 0$. The cost matrices are

$$Q_L = I_4, R_L = 1, \ S_L = 0.01 \cdot I_4$$

We do not require any control constraint (Q_u does not appear).

The definition of the disturbance $w(\Phi)$ and E are motivated by numerical reasons. The choice of S_L is also intuitive. In (3.11) it can be seen, that no matter how we choose $r > 0$, there will be a small neighbourhood of the origin, where

$$\|w\|_{S_L}{}^2 \leq \|x\|^2 + \|u\|^2 = \|\zeta\|^2$$

holds true. Consequently, the inequality in the definition of Class Δ_I is satisfied only in a well defined but unknown neighbourhood $\mathcal{N}$. Thus, the inequality for the running cost

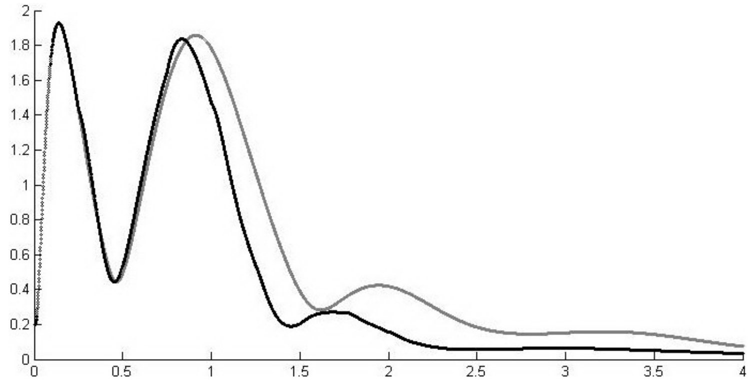
$$\mathcal{L}(z(t), w(t)) \geq 0$$

is not a priori guaranteed. However, after a successful simulation, a monotonically decreasing Lyapunov function indicates, that the system state did stay inside the above mentioned neighbourhood. If an increasing Lyapunov function is experienced, then the stabilization is not guaranteed and the feasibility can break down.

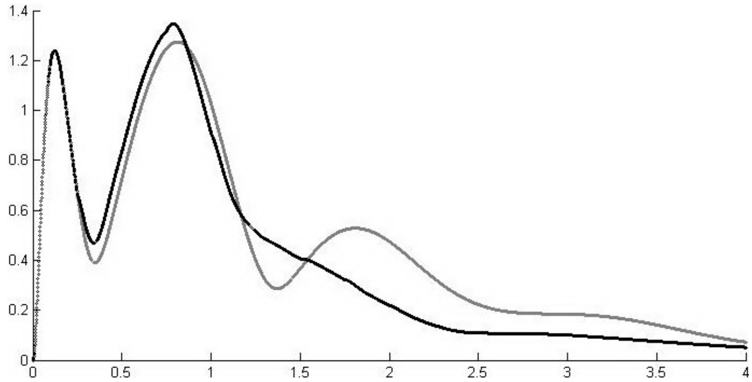
The auxiliary variable $\hat{\alpha}$ was set to $5 \cdot 10^{-3}$. We chose the initial state $x_0 = (0, 0, -0.2, 0)^\top$ and we simulated the system for 4 seconds, by dividing this interval into 16 equal time instances. The numerical problems, mentioned in Section 3.2, occured in some of the sampling instants. In these time instances we used the last successful solution of the LMI.

One can see the Lyapunov function on Figure 3.8(d), which is increasing, however the stabilization succeeded. By assigning a smaller multiplier term for S_L the Class Δ_I condition can be enforced. With such a vague knowledge about the nonlinearity, the stabilization cannot be guaranteed, however some successful simulations can be presented.

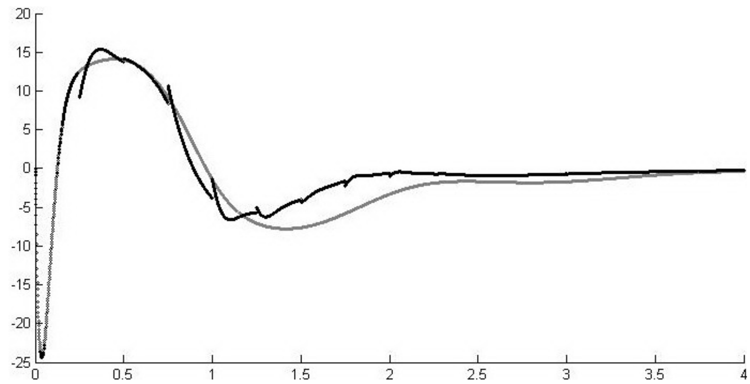
Figure 3.8: Simulation of system 3.3



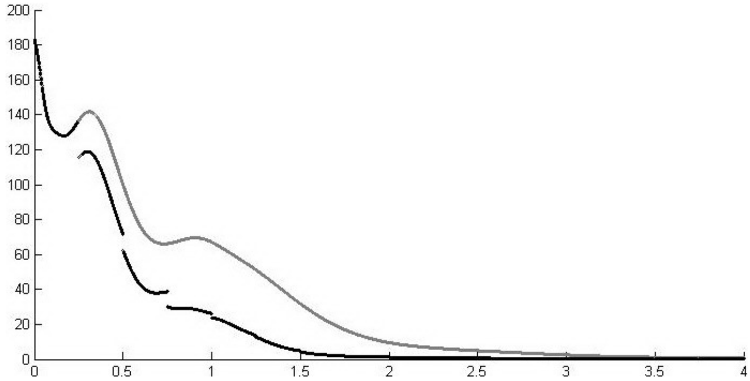
(a) $\|x(t)\|$



(b) $\|\hat{x}(t)\|$



(c) $u(t) = K_c^k \cdot \hat{x}(t)$



(d) $z(t)^\top P_k z(t)$

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