Finding Antiderivatives—Indefinite Integrals

Definitions

A function $F(x)$ is an **antiderivative** of a function $f(x)$ if

$$F'(x) = f(x)$$

for all x in the domain of f. The set of all antiderivatives of f is the **indefinite integral** of f with respect to x, denoted by

$$\int f(x)\,dx.$$

The symbol $\int$ is an **integral sign.** The function f is the **integrand** of the integral and x is the **variable of integration.**

According to Corollary 2 of the Mean Value Theorem (Section 3.2), once we have found one antiderivative F of a function f, the other antiderivatives of f differ from F by a constant. We indicate this in integral notation in the following way:

$$\boxed{\int f(x)\,dx = F(x) + C. \qquad (1)}$$

The constant C is the **constant of integration** or **arbitrary constant.** Equation (1) is read, "The indefinite integral of f with respect to x is $F(x) + C$." When we find $F(x) + C$, we say that we have **integrated** f and **evaluated** the integral.

EXAMPLE 1 Evaluate $\int 2x\,dx$.

Solution

$$\int 2x\,dx = \overbrace{x^2}^{\text{an antiderivative of } 2x} + \underbrace{C}_{\text{the arbitrary constant}}$$

The formula $x^2 + C$ generates all the antiderivatives of the function $2x$. The functions $x^2 + 1$, $x^2 - \pi$, and $x^2 + \sqrt{2}$ are all antiderivatives of the function $2x$, as you can check by differentiation. ❑

Many of the indefinite integrals needed in scientific work are found by reversing derivative formulas. You will see what we mean if you look at Table 4.1, which lists a number of standard integral forms side by side with their derivative-formula sources.

In case you are wondering why the integrals of the tangent, cotangent, secant, and cosecant do not appear in the table, the answer is that the usual formulas for them require logarithms. In Section 4.7, we will see that these functions do have antiderivatives, but we will have to wait until Chapters 6 and 7 to see what they are.

Table 4.1 Integral formulas

Indefinite integral	Reversed derivative formula
1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1,\ n$ rational	$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$
$\int dx = \int 1\, dx = x + C$ (special case)	$\frac{d}{dx}(x) = 1$
2. $\int \sin kx\, dx = -\frac{\cos kx}{k} + C$	$\frac{d}{dx}\left(-\frac{\cos kx}{k}\right) = \sin kx$
3. $\int \cos kx\, dx = \frac{\sin kx}{k} + C$	$\frac{d}{dx}\left(\frac{\sin kx}{k}\right) = \cos kx$
4. $\int \sec^2 x\, dx = \tan x + C$	$\frac{d}{dx} \tan x = \sec^2 x$
5. $\int \csc^2 x\, dx = -\cot x + C$	$\frac{d}{dx}(-\cot x) = \csc^2 x$
6. $\int \sec x \tan x\, dx = \sec x + C$	$\frac{d}{dx} \sec x = \sec x \tan x$
7. $\int \csc x \cot x\, dx = -\csc x + C$	$\frac{d}{dx}(-\csc x) = \csc x \cot x$

EXAMPLE 2 *Selected integrals from Table 4.1*

a) $$\int x^5\, dx = \frac{x^6}{6} + C$$ Formula 1 with $n = 5$

b) $$\int \frac{1}{\sqrt{x}}\, dx = \int x^{-1/2}\, dx = 2x^{1/2} + C = 2\sqrt{x} + C$$ Formula 1 with $n = -1/2$

c) $$\int \sin 2x\, dx = -\frac{\cos 2x}{2} + C$$ Formula 2 with $k = 2$

d) $$\int \cos \frac{x}{2}\, dx = \int \cos \frac{1}{2}x\, dx = \frac{\sin (1/2)x}{1/2} + C = 2 \sin \frac{x}{2} + C$$ Formula 3 with $k = 1/2$

❑

Finding an integral formula can sometimes be difficult, but checking it, once found, is relatively easy: differentiate the right-hand side. The derivative should be the integrand.

EXAMPLE 3

Right: $$\int x \cos x\, dx = x \sin x + \cos x + C$$

Reason: The derivative of the right-hand side is the integrand:

$$\frac{d}{dx}(x \sin x + \cos x + C) = x \cos x + \sin x - \sin x + 0 = x \cos x.$$

Wrong: $\int x \cos x \, dx = x \sin x + C$

Reason: The derivative of the right-hand side is not the integrand:

$$\frac{d}{dx}(x \sin x + C) = x \cos x + \sin x + 0 \neq x \cos x.$$

❑

Do not worry about how to derive the correct integral formula in Example 3. We will present a technique for doing so in Chapter 7.

Rules of Algebra for Antiderivatives

Among the things we know about antiderivatives are these:

1. A function is an antiderivative of a constant multiple kf of a function f if and only if it is k times an antiderivative of f.
2. In particular, a function is an antiderivative of $-f$ if and only if it is the negative of an antiderivative of f.
3. A function is an antiderivative of a sum or difference $f \pm g$ if and only if it is the sum or difference of an antiderivative of f and an antiderivative of g.

When we express these observations in integral notation, we get the standard arithmetic rules for indefinite integration (Table 4.2).

Table 4.2 Rules for indefinite integration

1. *Constant Multiple Rule:* $\int kf(x)\,dx = k\int f(x)\,dx$

   (Does not work if k varies with x.)

2. *Rule for Negatives:* $\int -f(x)\,dx = -\int f(x)\,dx$

   (Rule 1 with $k = -1$)

3. *Sum and Difference Rule:* $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

EXAMPLE 4 *Rewriting the constant of integration*

$\int 5 \sec x \tan x \, dx = 5 \int \sec x \tan x \, dx$	Table 4.2, Rule 1
$= 5(\sec x + C)$	Table 4.1, Formula 6
$= 5 \sec x + 5C$	First form
$= 5 \sec x + C'$	Shorter form, where C' is $5C$
$= 5 \sec x + C$	Usual form—no prime. Since 5 times an arbitrary constant is an arbitrary constant, we rename C'.

❑

What about all the different forms in Example 4? Each one gives all the antiderivatives of $f(x) = 5 \sec x \tan x$, so each answer is correct. But the least complicated of the three, and the usual choice, is

$$\int 5 \sec x \tan x \, dx = 5 \sec x + C.$$

Just as the Sum and Difference Rule for differentiation enables us to differentiate expressions term by term, the Sum and Difference Rule for integration enables us to integrate expressions term by term. When we do so, we combine the individual constants of integration into a single arbitrary constant at the end.

EXAMPLE 5 *Term-by-term integration*

Evaluate

$$\int (x^2 - 2x + 5)\, dx.$$

Solution If we recognize that $(x^3/3) - x^2 + 5x$ is an antiderivative of $x^2 - 2x + 5$, we can evaluate the integral as

$$\int (x^2 - 2x + 5)\, dx = \overbrace{\frac{x^3}{3} - x^2 + 5x}^{\text{antiderivative}} + \underbrace{C}_{\text{arbitrary constant}}.$$

If we do not recognize the antiderivative right away, we can generate it term by term with the Sum and Difference Rule:

$$\begin{aligned}\int (x^2 - 2x + 5)\, dx &= \int x^2\, dx - \int 2x\, dx + \int 5\, dx \\ &= \frac{x^3}{3} + C_1 - x^2 + C_2 + 5x + C_3.\end{aligned}$$

This formula is more complicated than it needs to be. If we combine C_1, C_2, and C_3 into a single constant $C = C_1 + C_2 + C_3$, the formula simplifies to

$$\frac{x^3}{3} - x^2 + 5x + C$$

and *still* gives all the antiderivatives there are. For this reason we recommend that you go right to the final form even if you elect to integrate term by term. Write

$$\begin{aligned}\int (x^2 - 2x + 5)\, dx &= \int x^2\, dx - \int 2x\, dx + \int 5\, dx \\ &= \frac{x^3}{3} - x^2 + 5x + C.\end{aligned}$$

Find the simplest antiderivative you can for each part and add the constant at the end. ❑

The Integrals of $\sin^2 x$ and $\cos^2 x$

We can sometimes use trigonometric identities to transform integrals we do not know how to evaluate into integrals we do know how to evaluate. The integral formulas for $\sin^2 x$ and $\cos^2 x$ arise frequently in applications.

EXAMPLE 6

a) $$\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx \qquad \sin^2 x = \frac{1-\cos 2x}{2}$$
$$= \frac{1}{2}\int (1-\cos 2x)\,dx = \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x\,dx$$
$$= \frac{1}{2}x - \frac{1}{2}\frac{\sin 2x}{2} + C = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

b) $$\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$
$$= \frac{x}{2} + \frac{\sin 2x}{4} + C$$ As in part (a), but with a sign change ❑

Exercises 4.1

Finding Antiderivatives

In Exercises 1–18, find an antiderivative for each function. Do as many as you can mentally. Check your answers by differentiation.

1. a) $2x$ **b)** x^2 **c)** $x^2 - 2x + 1$

2. a) $6x$ **b)** x^7 **c)** $x^7 - 6x + 8$

3. a) $-3x^{-4}$ **b)** x^{-4} **c)** $x^{-4} + 2x + 3$

4. a) $2x^{-3}$ **b)** $\frac{x^{-3}}{2} + x^2$ **c)** $-x^{-3} + x - 1$

5. a) $\frac{1}{x^2}$ **b)** $\frac{5}{x^2}$ **c)** $2 - \frac{5}{x^2}$

6. a) $-\frac{2}{x^3}$ **b)** $\frac{1}{2x^3}$ **c)** $x^3 - \frac{1}{x^3}$

7. a) $\frac{3}{2}\sqrt{x}$ **b)** $\frac{1}{2\sqrt{x}}$ **c)** $\sqrt{x} + \frac{1}{\sqrt{x}}$

8. a) $\frac{4}{3}\sqrt[3]{x}$ **b)** $\frac{1}{3\sqrt[3]{x}}$ **c)** $\sqrt[3]{x} + \frac{1}{\sqrt[3]{x}}$

9. a) $\frac{2}{3}x^{-1/3}$ **b)** $\frac{1}{3}x^{-2/3}$ **c)** $-\frac{1}{3}x^{-4/3}$

10. a) $\frac{1}{2}x^{-1/2}$ **b)** $-\frac{1}{2}x^{-3/2}$ **c)** $-\frac{3}{2}x^{-5/2}$

11. a) $-\pi \sin \pi x$ **b)** $3 \sin x$ **c)** $\sin \pi x - 3 \sin 3x$

12. a) $\pi \cos \pi x$ **b)** $\frac{\pi}{2}\cos\frac{\pi x}{2}$ **c)** $\cos\frac{\pi x}{2} + \pi \cos x$

13. a) $\sec^2 x$ **b)** $\frac{2}{3}\sec^2\frac{x}{3}$ **c)** $-\sec^2\frac{3x}{2}$

14. a) $\csc^2 x$ **b)** $-\frac{3}{2}\csc^2\frac{3x}{2}$ **c)** $1 - 8\csc^2 2x$

15. a) $\csc x \cot x$ **b)** $-\csc 5x \cot 5x$ **c)** $-\pi \csc\frac{\pi x}{2}\cot\frac{\pi x}{2}$

16. a) $\sec x \tan x$ **b)** $4 \sec 3x \tan 3x$ **c)** $\sec\frac{\pi x}{2}\tan\frac{\pi x}{2}$

17. $(\sin x - \cos x)^2$

18. $(1 + 2\cos x)^2$

Evaluating Integrals

Evaluate the integrals in Exercises 19–58. Check your answers by differentiation.

19. $\int (x+1)\,dx$

20. $\int (5-6x)\,dx$

21. $\int \left(3t^2 + \frac{t}{2}\right) dt$

22. $\int \left(\frac{t^2}{2} + 4t^3\right) dt$

23. $\int (2x^3 - 5x + 7)\,dx$

24. $\int (1 - x^2 - 3x^5)\,dx$

25. $\int \left(\frac{1}{x^2} - x^2 - \frac{1}{3}\right) dx$

26. $\int \left(\frac{1}{5} - \frac{2}{x^3} + 2x\right) dx$

27. $\int x^{-1/3}\,dx$

28. $\int x^{-5/4}\,dx$

29. $\int \left(\sqrt{x} + \sqrt[3]{x}\right) dx$

30. $\int \left(\frac{\sqrt{x}}{2} + \frac{2}{\sqrt{x}}\right) dx$

31. $\int \left(8y - \frac{2}{y^{1/4}}\right) dy$

32. $\int \left(\frac{1}{7} - \frac{1}{y^{5/4}}\right) dy$

33. $\int 2x\left(1 - x^{-3}\right) dx$

34. $\int x^{-3}(x+1)\, dx$

35. $\int \frac{t\sqrt{t} + \sqrt{t}}{t^2}\, dt$

36. $\int \frac{4 + \sqrt{t}}{t^3}\, dt$

37. $\int (-2 \cos t)\, dt$

38. $\int (-5 \sin t)\, dt$

39. $\int 7 \sin \frac{\theta}{3}\, d\theta$

40. $\int 3 \cos 5\theta\, d\theta$

41. $\int (-3 \csc^2 x)\, dx$

42. $\int \left(-\frac{\sec^2 x}{3}\right) dx$

43. $\int \frac{\csc \theta \cot \theta}{2}\, d\theta$

44. $\int \frac{2}{5} \sec \theta \tan \theta\, d\theta$

45. $\int (4 \sec x \tan x - 2 \sec^2 x)\, dx$

46. $\int \frac{1}{2} (\csc^2 x - \csc x \cot x)\, dx$

47. $\int (\sin 2x - \csc^2 x)\, dx$

48. $\int (2 \cos 2x - 3 \sin 3x)\, dx$

49. $\int 4 \sin^2 y\, dy$

50. $\int \frac{\cos^2 y}{7}\, dy$

51. $\int \frac{1 + \cos 4t}{2}\, dt$

52. $\int \frac{1 - \cos 6t}{2}\, dt$

53. $\int (1 + \tan^2 \theta)\, d\theta$

(*Hint:* $1 + \tan^2 \theta = \sec^2 \theta$)

54. $\int (2 + \tan^2 \theta)\, d\theta$

55. $\int \cot^2 x\, dx$

(*Hint:* $1 + \cot^2 x = \csc^2 x$)

56. $\int (1 - \cot^2 x)\, dx$

57. $\int \cos \theta (\tan \theta + \sec \theta)\, d\theta$

58. $\int \frac{\csc \theta}{\csc \theta - \sin \theta}\, d\theta$

Checking Integration Formulas

Verify the integral formulas in Exercises 59–64 by differentiation. In Section 4.3, we will see where formulas like these come from.

59. $\int (7x - 2)^3\, dx = \frac{(7x-2)^4}{28} + C$

60. $\int (3x + 5)^{-2}\, dx = -\frac{(3x+5)^{-1}}{3} + C$

61. $\int \sec^2(5x - 1)\, dx = \frac{1}{5} \tan(5x - 1) + C$

62. $\int \csc^2\left(\frac{x-1}{3}\right) dx = -3 \cot\left(\frac{x-1}{3}\right) + C$

63. $\int \frac{1}{(x+1)^2}\, dx = -\frac{1}{x+1} + C$

64. $\int \frac{1}{(x+1)^2}\, dx = \frac{x}{x+1} + C$

65. Right, or wrong? Say which for each formula and give a brief reason for each answer.

a) $\int x \sin x\, dx = \frac{x^2}{2} \sin x + C$

b) $\int x \sin x\, dx = -x \cos x + C$

c) $\int x \sin x\, dx = -x \cos x + \sin x + C$

66. Right, or wrong? Say which for each formula and give a brief reason for each answer.

a) $\int \tan \theta \sec^2 \theta\, d\theta = \frac{\sec^3 \theta}{3} + C$

b) $\int \tan \theta \sec^2 \theta\, d\theta = \frac{1}{2} \tan^2 \theta + C$

c) $\int \tan \theta \sec^2 \theta\, d\theta = \frac{1}{2} \sec^2 \theta + C$

67. Right, or wrong? Say which for each formula and give a brief reason for each answer.

a) $\int (2x+1)^2\, dx = \frac{(2x+1)^3}{3} + C$

b) $\int 3(2x+1)^2\, dx = (2x+1)^3 + C$

c) $\int 6(2x+1)^2\, dx = (2x+1)^3 + C$

68. Right, or wrong? Say which for each formula and give a brief reason for each answer.

a) $\int \sqrt{2x+1}\, dx = \sqrt{x^2 + x + C}$

b) $\int \sqrt{2x+1}\, dx = \sqrt{x^2 + x} + C$

c) $\int \sqrt{2x+1}\, dx = \frac{1}{3}\left(\sqrt{2x+1}\right)^3 + C$

Theory and Examples

69. Suppose that

$$f(x) = \frac{d}{dx}(1 - \sqrt{x}) \quad \text{and} \quad g(x) = \frac{d}{dx}(x + 2).$$

Find:

a) $\int f(x)\, dx$

b) $\int g(x)\, dx$

c) $\int [-f(x)]\, dx$

d) $\int [-g(x)]\, dx$

e) $\int [f(x) + g(x)]\, dx$

f) $\int [f(x) - g(x)]\, dx$

This same equation, from another point of view, says that $u^{n+1}/(n+1)$ is one of the antiderivatives of the function $u^n(du/dx)$. Therefore,

$$\int \left(u^n \frac{du}{dx}\right) dx = \frac{u^{n+1}}{n+1} + C.$$

The integral on the left-hand side of this equation is usually written in the simpler "differential" form,

$$\int u^n du,$$

obtained by treating the dx's as differentials that cancel. Combining the last two equations gives the following rule.

Equation (1) actually holds for any real exponent $n \neq -1$, as we will see in Chapter 6.

> If u is any differentiable function,
>
> $$\int u^n du = \frac{u^{n+1}}{n+1} + C. \qquad (n \neq -1,\ n \text{ rational}) \tag{1}$$

In deriving Eq. (1) we assumed u to be a differentiable function of the variable x, but the name of the variable does not matter and does not appear in the final formula. We could have represented the variable with θ, t, y, or any other letter. Equation (1) says that whenever we can cast an integral in the form

$$\int u^n du, \qquad (n \neq -1)$$

with u a differentiable function and du its differential, we can evaluate the integral as $\left[u^{n+1}/(n+1)\right] + C$.

EXAMPLE 1 Evaluate $\displaystyle\int (x+2)^5\, dx$.

Solution We can put the integral in the form

$$\int u^n du$$

by substituting

$$u = x + 2, \qquad du = d(x+2) = \frac{d}{dx}(x+2) \cdot dx$$
$$= 1 \cdot dx = dx.$$

Then

$$\int (x+2)^5 dx = \int u^5 du \qquad u = x+2, \quad du = dx$$

$$= \frac{u^6}{6} + C \qquad \text{Integrate, using Eq. (1) with } n = 5.$$

$$= \frac{(x+2)^6}{6} + C. \qquad \text{Replace } u \text{ by } x+2.$$

❑

EXAMPLE 2

$$\int \sqrt{1+y^2} \cdot 2y\,dy = \int u^{1/2}\,du \qquad \text{Let } u = 1+y^2,\ du = 2y\,dy.$$

$$= \frac{u^{(1/2)+1}}{(1/2)+1} + C \qquad \text{Integrate, using Eq. (1) with } n = 1/2.$$

$$= \frac{2}{3}u^{3/2} + C \qquad \text{Simpler form}$$

$$= \frac{2}{3}(1+y^2)^{3/2} + C \qquad \text{Replace } u \text{ by } 1+y^2.$$

□

EXAMPLE 3 *Adjusting the integrand by a constant*

$$\int \sqrt{4t-1}\,dt = \int u^{1/2} \cdot \frac{1}{4}du \qquad \text{Let } u = 4t-1,\ du = 4\,dt,\ (1/4)\,du = dt.$$

$$= \frac{1}{4}\int u^{1/2}du \qquad \text{With the } 1/4 \text{ out front, the integral is now in standard form.}$$

$$= \frac{1}{4} \cdot \frac{u^{3/2}}{3/2} + C \qquad \text{Integrate, using Eq. (1) with } n = 1/2.$$

$$= \frac{1}{6}u^{3/2} + C \qquad \text{Simpler form}$$

$$= \frac{1}{6}(4t-1)^{3/2} + C \qquad \text{Replace } u \text{ by } 4t-1.$$

□

Trigonometric Functions

If u is a differentiable function of x, then $\sin u$ is a differentiable function of x. The Chain Rule gives the derivative of $\sin u$ as

$$\frac{d}{dx}\sin u = \cos u\,\frac{du}{dx}.$$

From another point of view, however, this same equation says that $\sin u$ is one of the antiderivatives of the product $\cos u \cdot (du/dx)$. Therefore,

$$\int \left(\cos u\,\frac{du}{dx}\right)dx = \sin u + C.$$

A formal cancellation of the dx's in the integral on the left leads to the following rule.

> If u is a differentiable function, then
>
> $$\int \cos u\,du = \sin u + C. \qquad (2)$$

Equation (2) says that whenever we can cast an integral in the form

$$\int \cos u\, du,$$

we can integrate with respect to u to evaluate the integral as $\sin u + C$.

EXAMPLE 4

$\int \cos(7\theta + 5)\, d\theta$	$= \int \cos u \cdot \frac{1}{7} du$	Let $u = 7\theta + 5$, $du = 7\, d\theta$, $(1/7)\, du = d\theta$.
	$= \frac{1}{7} \int \cos u\, du$	With (1/7) out front, the integral is now in standard form.
	$= \frac{1}{7} \sin u + C$	Integrate with respect to u.
	$= \frac{1}{7} \sin(7\theta + 5) + C$	Replace u by $7\theta + 5$. ❑

The companion formula for the integral of $\sin u$ when u is a differentiable function is

$$\int \sin u\, du = -\cos u + C. \qquad (3)$$

EXAMPLE 5

$\int x^2 \sin(x^3)\, dx$	$= \int \sin(x^3) \cdot x^2 dx$	
	$= \int \sin u \cdot \frac{1}{3} du$	Let $u = x^3$ $du = 3x^2\, dx$ $(1/3)du = x^2\, dx$.
	$= \frac{1}{3} \int \sin u\, du$	
	$= \frac{1}{3}(-\cos u) + C$	Integrate with respect to u.
	$= -\frac{1}{3} \cos(x^3) + C$	Replace u by x^3. ❑

The Chain Rule formulas for the derivatives of the tangent, cotangent, secant, and cosecant of a differentiable function u lead to the following integrals.

$$\int \sec^2 u\, du = \tan u + C \qquad (4)$$

$$\int \csc^2 u\, du = -\cot u + C \qquad (5)$$

$$\int \sec u \tan u\, du = \sec u + C \qquad (6)$$

$$\int \csc u \cot u\, du = -\csc u + C \qquad (7)$$

In each formula, u is a differentiable function of a real variable. Each formula can be checked by differentiating the right-hand side with respect to that variable. In each case, the Chain Rule applies to produce the integrand on the left.

EXAMPLE 6

$$\begin{aligned}
\int \frac{1}{\cos^2 2\theta}\,d\theta &= \int \sec^2 2\theta\,d\theta && \sec 2\theta = \frac{1}{\cos 2\theta}\\
&= \int \sec^2 u \cdot \frac{1}{2}\,du && \text{Let } u = 2\theta,\ du = 2\,d\theta,\ d\theta = (1/2)\,du.\\
&= \frac{1}{2}\int \sec^2 u\,du\\
&= \frac{1}{2}\tan u + C && \text{Integrate, using Eq. (4).}\\
&= \frac{1}{2}\tan 2\theta + C && \text{Replace } u \text{ by } 2\theta.
\end{aligned}$$

Check:

$$\begin{aligned}
\frac{d}{d\theta}\left(\frac{1}{2}\tan 2\theta + C\right) &= \frac{1}{2}\cdot\frac{d}{d\theta}(\tan 2\theta) + 0\\
&= \frac{1}{2}\cdot\left(\sec^2 2\theta \cdot \frac{d}{d\theta}(2\theta)\right) && \text{Chain Rule}\\
&= \frac{1}{2}\cdot \sec^2 2\theta \cdot 2 = \frac{1}{\cos^2 2\theta}.
\end{aligned}$$

❑

The Substitution Method of Integration

The substitutions in the preceding examples are all instances of the following general rule.

$$\begin{aligned}
\int f(g(x))\cdot g'(x)\,dx &= \int f(u)\,du && \text{1. Substitute } u = g(x),\ du = g'(x)\,dx.\\
&= F(u) + C && \text{2. Evaluate by finding an antiderivative } F(u) \text{ of } f(u). \text{ (Any one will do.)}\\
&= F(g(x)) + C && \text{3. Replace } u \text{ by } g(x).
\end{aligned}$$

These three steps are the steps of the substitution method of integration. The method works because $F(g(x))$ is an antiderivative of $f(g(x))\cdot g'(x)$ whenever F is an antiderivative of f:

$$\begin{aligned}
\frac{d}{dx}F(g(x)) &= F'(g(x))\cdot g'(x) && \text{Chain Rule}\\
&= f(g(x))\cdot g'(x) && \text{Because } F' = f
\end{aligned}$$

Implicit in the substitution method is the assumption that we are replacing x by a function of u. Thus, the substitution $u = g(x)$ must be solvable for x to give x as a function $x = g^{-1}(u)$ ("g inverse of u"). The domains of u and x may need to be restricted on occasion to make this possible. You need not be concerned with this issue at the moment. We will discuss inverses in Section 6.1 and treat the theory of substitutions in greater detail in Sections 7.4 and 13.7.

The Substitution Method of Integration

Take these steps to evaluate the integral

$$\int f(g(x))g'(x)\,dx,$$

when f and g' are continuous functions:

Step 1: Substitute $u = g(x)$ and $du = g'(x)\,dx$ to obtain the integral

$$\int f(u)\,du.$$

Step 2: Integrate with respect to u.
Step 3: Replace u by $g(x)$ in the result.

EXAMPLE 7

$$\int (x^2 + 2x - 3)^2 (x + 1)\, dx = \int u^2 \cdot \frac{1}{2}\, du$$

Let $u = x^2 + 2x - 3$, $du = 2x\,dx + 2\,dx = 2(x+1)\,dx$, $(1/2)\,du = (x+1)\,dx$.

$$= \frac{1}{2}\int u^2\, du$$

$$= \frac{1}{2} \cdot \frac{u^3}{3} + C = \frac{1}{6}u^3 + C$$

Integrate with respect to u.

$$= \frac{1}{6}(x^2 + 2x - 3)^3 + C$$

Replace u. ❑

EXAMPLE 8

$$\int \sin^4 t \cos t\, dt = \int u^4\, du$$

Let $u = \sin t$, $du = \cos t\, dt$.

$$= \frac{u^5}{5} + C$$

Integrate with respect to u.

$$= \frac{\sin^5 t}{5} + C$$

Replace u. ❑

The success of the substitution method depends on finding a substitution that will change an integral we cannot evaluate directly into one that we can. If the first substitution fails, we can try to simplify the integrand further with an additional substitution or two. (You will see what we mean if you do Exercises 47 and 48.) Alternatively, we can start afresh. There can be more than one good way to start, as in the next example.

EXAMPLE 9 Evaluate

$$\int \frac{2z\, dz}{\sqrt[3]{z^2 + 1}}.$$

Solution We can use the substitution method of integration as an exploratory tool: substitute for the most troublesome part of the integrand and see how things work out. For the integral here, we might try $u = z^2 + 1$ or we might even press our luck and take u to be the entire cube root. Here is what happens in each case.

Solution 1 Substitute $u = z^2 + 1$.

$$\int \frac{2z\, dz}{\sqrt[3]{z^2 + 1}} = \int \frac{du}{u^{1/3}}$$

Let $u = z^2 + 1$, $du = 2z\, dz$.

$$= \int u^{-1/3}\, du$$

In the form $\int u^n\, du$

$$= \frac{u^{2/3}}{2/3} + C$$

Integrate with respect to u.

$$= \frac{3}{2}u^{2/3} + C$$

$$= \frac{3}{2}(z^2 + 1)^{2/3} + C$$

Replace u by $z^2 + 1$.

Solution 2 Substitute $u = \sqrt[3]{z^2+1}$ instead.

$$\int \frac{2z\,dz}{\sqrt[3]{z^2+1}} = \int \frac{3u^2\,du}{u}$$

Let $u = \sqrt[3]{z^2+1}$, $u^3 = z^2+1$, $3u^2\,du = 2z\,dz$.

$$= 3\int u\,du$$

$$= 3 \cdot \frac{u^2}{2} + C$$

Integrate with respect to u.

$$= \frac{3}{2}(z^2+1)^{2/3} + C$$

Replace u by $(z^2+1)^{1/3}$.

❑

Exercises 4.3

Evaluating Integrals

Evaluate the indefinite integrals in Exercises 1–12 by using the given substitutions to reduce the integrals to standard form.

1. $\int \sin 3x\,dx, \quad u = 3x$

2. $\int x \sin(2x^2)\,dx, \quad u = 2x^2$

3. $\int \sec 2t \tan 2t\,dt, \quad u = 2t$

4. $\int \left(1 - \cos\frac{t}{2}\right)^2 \sin\frac{t}{2}\,dt, \quad u = 1 - \cos\frac{t}{2}$

5. $\int 28(7x-2)^{-5}\,dx, \quad u = 7x - 2$

6. $\int x^3(x^4-1)^2\,dx, \quad u = x^4 - 1$

7. $\int \frac{9r^2\,dr}{\sqrt{1-r^3}}, \quad u = 1 - r^3$

8. $\int 12(y^4+4y^2+1)^2(y^3+2y)\,dy, \quad u = y^4+4y^2+1$

9. $\int \sqrt{x}\sin^2(x^{3/2}-1)\,dx, \quad u = x^{3/2} - 1$

10. $\int \frac{1}{x^2}\cos^2\left(\frac{1}{x}\right)dx, \quad u = -\frac{1}{x}$

11. $\int \csc^2 2\theta \cot 2\theta\,d\theta$

a) Using $u = \cot 2\theta$ **b)** Using $u = \csc 2\theta$

12. $\int \frac{dx}{\sqrt{5x+8}}$

a) Using $u = 5x + 8$ **b)** Using $u = \sqrt{5x+8}$

Evaluate the integrals in Exercises 13–46.

13. $\int \sqrt{3-2s}\,ds$

14. $\int (2x+1)^3\,dx$

15. $\int \frac{1}{\sqrt{5s+4}}\,ds$

16. $\int \frac{3\,dx}{(2-x)^2}$

17. $\int \theta\sqrt[4]{1-\theta^2}\,d\theta$

18. $\int 8\theta\sqrt[3]{\theta^2-1}\,d\theta$

19. $\int 3y\sqrt{7-3y^2}\,dy$

20. $\int \frac{4y\,dy}{\sqrt{2y^2+1}}$

21. $\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\,dx$

22. $\int \frac{(1+\sqrt{x})^3}{\sqrt{x}}\,dx$

23. $\int \cos(3z+4)\,dz$

24. $\int \sin(8z-5)\,dz$

25. $\int \sec^2(3x+2)\,dx$

26. $\int \tan^2 x \sec^2 x\,dx$

27. $\int \sin^5\frac{x}{3}\cos\frac{x}{3}\,dx$

28. $\int \tan^7\frac{x}{2}\sec^2\frac{x}{2}\,dx$

29. $\int r^2\left(\frac{r^3}{18}-1\right)^5 dr$

30. $\int r^4\left(7-\frac{r^5}{10}\right)^3 dr$

31. $\int x^{1/2}\sin(x^{3/2}+1)\,dx$

32. $\int x^{1/3}\sin(x^{4/3}-8)\,dx$

33. $\int \sec\left(v+\frac{\pi}{2}\right)\tan\left(v+\frac{\pi}{2}\right)dv$

34. $\int \csc\left(\frac{v-\pi}{2}\right)\cot\left(\frac{v-\pi}{2}\right)dv$

35. $\int \frac{\sin(2t+1)}{\cos^2(2t+1)}\,dt$

36. $\int \frac{6\cos t}{(2+\sin t)^3}\,dt$

37. $\int \sqrt{\cot y}\,\csc^2 y\,dy$

38. $\int \frac{\sec z\tan z}{\sqrt{\sec z}}\,dz$

39. $\int \frac{1}{t^2}\cos\left(\frac{1}{t}-1\right)dt$

40. $\int \frac{1}{\sqrt{t}}\cos(\sqrt{t}+3)\,dt$

41. $\int \frac{1}{\theta^2}\sin\frac{1}{\theta}\cos\frac{1}{\theta}\,d\theta$

42. $\int \frac{\cos\sqrt{\theta}}{\sqrt{\theta}\sin^2\sqrt{\theta}}\,d\theta$

43. $\int (s^3+2s^2-5s+5)(3s^2+4s-5)\,ds$

44. $\int (\theta^4-2\theta^2+8\theta-2)(\theta^3-\theta+2)\,d\theta$

45. $\int t^3(1+t^4)^3\,dt$

46. $\int \sqrt{\frac{x-1}{x^5}}\,dx$

Simplifying Integrals Step by Step

If you do not know what substitution to make, try reducing the integral step by step, using a trial substitution to simplify the integral a bit and then another to simplify it some more. You will see what we mean if you try the sequences of substitutions in Exercises 47 and 48.

47. $\int \frac{18\tan^2 x\sec^2 x}{(2+\tan^3 x)^2}\,dx$

a) $u=\tan x$, followed by $v=u^3$, then by $w=2+v$
b) $u=\tan^3 x$, followed by $v=2+u$
c) $u=2+\tan^3 x$

48. $\int \sqrt{1+\sin^2(x-1)}\,\sin(x-1)\cos(x-1)\,dx$

a) $u=x-1$, followed by $v=\sin u$, then by $w=1+v^2$
b) $u=\sin(x-1)$, followed by $v=1+u^2$
c) $u=1+\sin^2(x-1)$

Evaluate the integrals in Exercises 49 and 50.

49. $\int \frac{(2r-1)\cos\sqrt{3(2r-1)^2+6}}{\sqrt{3(2r-1)^2+6}}\,dr$

50. $\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta}\cos^3\sqrt{\theta}}\,d\theta$

Initial Value Problems

Solve the initial value problems in Exercises 51–56.

51. $\frac{ds}{dt}=12t\,(3t^2-1)^3,\quad s(1)=3$

52. $\frac{dy}{dx}=4x\,(x^2+8)^{-1/3},\quad y(0)=0$

53. $\frac{ds}{dt}=8\sin^2\left(t+\frac{\pi}{12}\right),\quad s(0)=8$

54. $\frac{dr}{d\theta}=3\cos^2\left(\frac{\pi}{4}-\theta\right),\quad r(0)=\frac{\pi}{8}$

55. $\frac{d^2s}{dt^2}=-4\sin\left(2t-\frac{\pi}{2}\right),\quad s'(0)=100,\ s(0)=0$

56. $\frac{d^2y}{dx^2}=4\sec^2 2x\tan 2x,\quad y'(0)=4,\ y(0)=-1$

57. The velocity of a particle moving back and forth on a line is $v=ds/dt=6\sin 2t$ m/sec for all t. If $s=0$ when $t=0$, find the value of s when $t=\pi/2$ sec.

58. The acceleration of a particle moving back and forth on a line is $a=d^2s/dt^2=\pi^2\cos\pi t$ m/sec^2 for all t. If $s=0$ and $v=8$ m/sec when $t=0$, find s when $t=1$ sec.

Theory and Examples

59. It looks as if we can integrate $2\sin x\cos x$ with respect to x in three different ways:

a) $\int 2\sin x\cos x\,dx=\int 2u\,du \qquad u=\sin x,$
$$=u^2+C_1=\sin^2 x+C_1$$

b) $\int 2\sin x\cos x\,dx=\int -2u\,du \qquad u=\cos x,$
$$=-u^2+C_2=-\cos^2 x+C_2$$

c) $\int 2\sin x\cos x\,dx=\int \sin 2x\,dx \qquad 2\sin x\cos x=\sin 2x$
$$=-\frac{\cos 2x}{2}+C_3.$$

Can all three integrations be correct? Give reasons for your answer.

60. The substitution $u=\tan x$ gives
$$\int \sec^2 x\tan x\,dx=\int u\,du=\frac{u^2}{2}+C=\frac{\tan^2 x}{2}+C.$$
The substitution $u=\sec x$ gives
$$\int \sec^2 x\tan x\,dx=\int u\,du=\frac{u^2}{2}+C=\frac{\sec^2 x}{2}+C.$$
Can both integrations be correct? Give reasons for your answer.