CHAPTER 3

Applications of Derivatives

OVERVIEW This chapter shows how to draw conclusions from derivatives. We use derivatives to find extreme values of functions, to predict and analyze the shapes of graphs, to find replacements for complicated formulas, to determine how sensitive formulas are to errors in measurement, and to find the zeros of functions numerically. The key to many of these accomplishments is the Mean Value Theorem, a theorem whose corollaries provide the gateway to integral calculus in Chapter 4.

3.1 Extreme Values of Functions

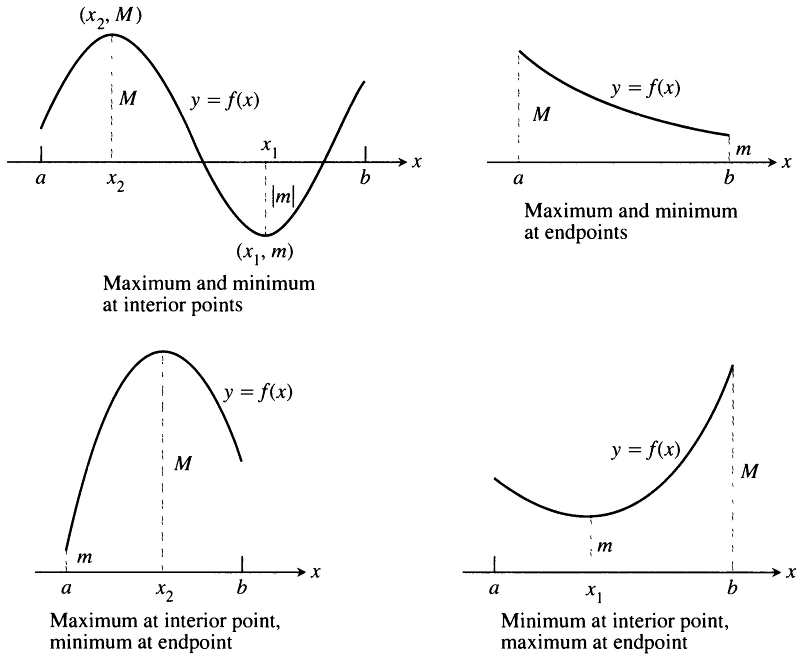
This section shows how to locate and identify extreme values of continuous functions.

The Max-Min Theorem

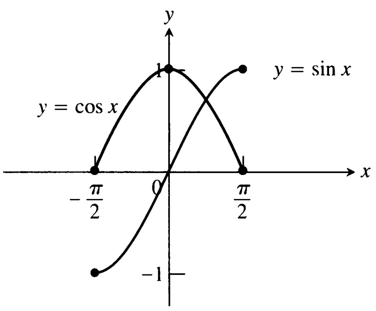
A function that is continuous at every point of a closed interval has an absolute maximum and an absolute minimum value on the interval. We always look for these values when we graph a function, and we will see the role they play in problem solving (this chapter) and in the development of the integral calculus (Chapters 4 and 5).

Theorem 1
The Max-Min Theorem for Continuous Functions
If f is continuous at every point of a closed interval I, then f assumes both an absolute maximum value M and an absolute minimum value m somewhere in I. That is, there are numbers x_1 and x_2 in I with $f(x_1) = m$, $f(x_2) = M$, and $m \leq f(x) \leq M$ for every other x in I (Fig. 3.1 on the following page).

The proof of Theorem 1 requires a detailed knowledge of the real number system and we will not give it here.



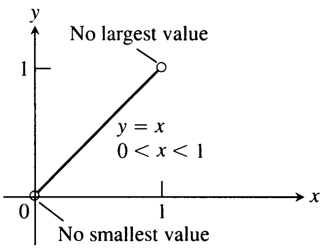
3.1 Typical arrangements of a continuous function's absolute maxima and minima on a closed interval $[a, b]$.



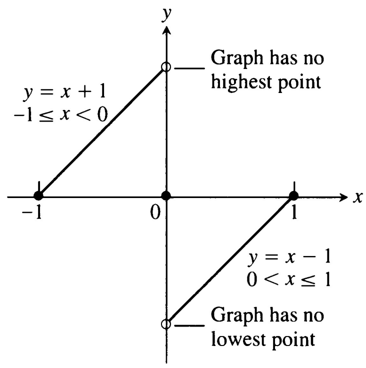
3.2 Figure for Example 1.

EXAMPLE 1 On $[-\pi/2, \pi/2]$, $f(x) = \cos x$ takes on a maximum value of 1 (once) and a minimum value of 0 (twice). The function $g(x) = \sin x$ takes on a maximum value of 1 and a minimum value of -1 (Fig. 3.2). ❑

As Figs. 3.3 and 3.4 show, the requirements that the interval be closed and the function continuous are key ingredients of Theorem 1. Without them, the conclusion of the theorem need not hold.



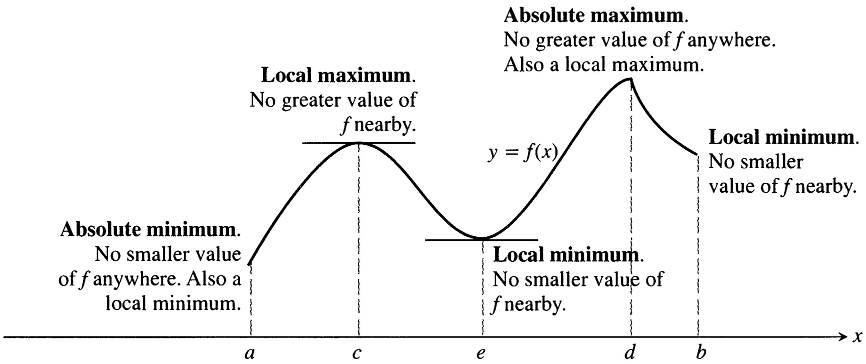
3.3 On an open interval, a continuous function need not have either a maximum or a minimum value. The function $f(x) = x$ has neither a largest nor a smallest value on $(0, 1)$.



3.4 Even a single point of discontinuity can keep a function from having either a maximum or a minimum value on a closed interval. The function

$$y = \begin{cases} x + 1, & -1 \le x < 0 \\ 0, & x = 0 \\ x - 1, & 0 < x \le 1 \end{cases}$$

is continuous at every point of $[-1,1]$ except $x = 0$, yet its graph over $[-1,1]$ has neither a highest nor a lowest point.



3.5 How to classify maxima and minima.

Local vs. Absolute (Global) Extrema

Figure 3.5 shows a graph with five extreme points. The function's absolute minimum occurs at a even though at e the function's value is smaller than at any other point *nearby.* The curve rises to the left and falls to the right around c, making $f(c)$ a maximum locally. The function attains its absolute maximum at d.

Definition
Absolute Extreme Values

Let f be a function with domain D. Then f has an **absolute maximum** value on D at a point c if

$$f(x) \leq f(c) \qquad \text{for all } x \text{ in } D$$

and an **absolute minimum** value on D at c if

$$f(x) \geq f(c) \qquad \text{for all } x \text{ in } D.$$

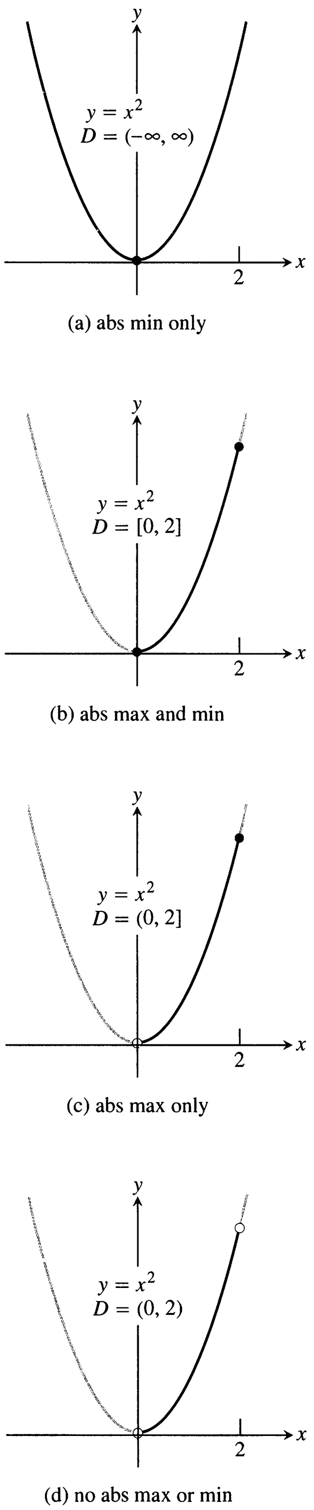
Absolute maximum and minimum values are called absolute **extrema** (plural of the Latin *extremum*). Absolute extrema are also called **global** extrema.

Functions with the same defining rule can have different extrema, depending on the domain.

EXAMPLE 2 (See Fig. 3.6.)

	Function rule	**Domain D**	**Absolute extrema on D (if any)**
a)	$y = x^2$	$(-\infty, \infty)$	No absolute maximum. Absolute minimum of 0 at $x = 0$.
b)	$y = x^2$	$[0, 2]$	Absolute maximum of $(2)^2 = 4$ at $x = 2$. Absolute minimum of 0 at $x = 0$.
c)	$y = x^2$	$(0, 2]$	Absolute maximum of 4 at $x = 2$. No absolute minimum.
d)	$y = x^2$	$(0, 2)$	No absolute extrema.

❑



3.6 Graphs for Example 2.

Definition
Local Extreme Values

A function f has a **local maximum** value at an interior point c of its domain if

$$f(x) \leq f(c) \qquad \text{for all } x \text{ in some open interval containing } c.$$

A function f has a **local minimum** value at an interior point c of its domain if

$$f(x) \geq f(c) \qquad \text{for all } x \text{ in some open interval containing } c.$$

We can extend the definitions of local extrema to the endpoints of intervals by defining f to have a **local maximum** or **local minimum** value *at an endpoint* c if the appropriate inequality holds for all x in some half-open interval in its domain containing c. In Fig. 3.5, the function f has local maxima at c and d and local minima at a, e, and b.

An absolute maximum is also a local maximum. Being the largest value overall, it is also the largest value in its immediate neighborhood. Hence, *a list of all local maxima will automatically include the absolute maximum if there is one*. Similarly, *a list of all local minima will include the absolute minimum if there is one.*

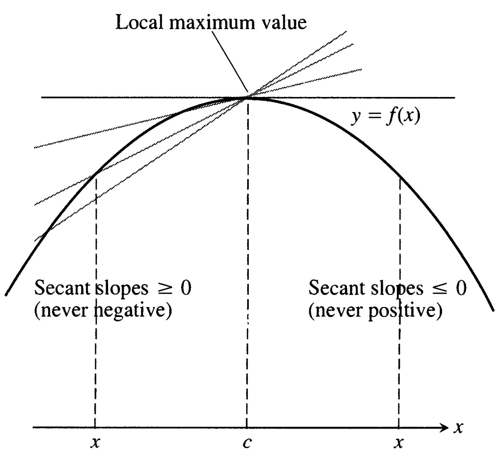
Finding Extrema

The next theorem explains why we usually need to investigate only a few values to find a function's extrema.

Theorem 2
The First Derivative Theorem for Local Extreme Values

If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then

$$f'(c) = 0.$$



3.7 A curve with a local maximum value. The slope at c, simultaneously the limit of nonpositive numbers and nonnegative numbers, is zero.

Proof To show that $f'(c)$ is zero at a local extremum, we show first that $f'(c)$ cannot be positive and second that $f'(c)$ cannot be negative. The only number that is neither positive nor negative is zero, so that is what $f'(c)$ must be.

To begin, suppose that f has a local maximum value at $x = c$ (Fig. 3.7) so that $f(x) - f(c) \leq 0$ for all values of x near enough to c. Since c is an interior point of f's domain, $f'(c)$ is defined by the two-sided limit

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

This means that the right-hand and left-hand limits both exist at $x = c$ and equal $f'(c)$. When we examine these limits separately, we find that

$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \leq 0. \qquad \text{Because } (x - c) > 0 \text{ and } f(x) \leq f(c) \qquad (1)$$

Similarly,

$$f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \geq 0. \qquad \text{Because } (x - c) < 0 \text{ and } f(x) \leq f(c) \tag{2}$$

Together, (1) and (2) imply $f'(c) = 0$.

This proves the theorem for local maximum values. To prove it for local minimum values, we simply use $f(x) \geq f(c)$, which reverses the inequalities in (1) and (2). ❑

Theorem 2 says that a function's first derivative is always zero at an interior point where the function has a local extreme value and the derivative is defined. Hence the only places where a function f can possibly have an extreme value (local or global) are

1. interior points where $f' = 0$,
2. interior points where f' is undefined,
3. endpoints of the domain of f.

The following definition helps us to summarize.

Definition

An interior point of the domain of a function f where f' is zero or undefined is a **critical point** of f.

Summary

The only domain points where a function can assume extreme values are critical points and endpoints.

Most quests for extreme values call for finding the absolute extrema of a continuous function on a closed interval. Theorem 1 assures us that such values exist; Theorem 2 tells us that they are taken on only at critical points and endpoints. These points are often so few in number that we can simply list them and calculate the corresponding function values to see what the largest and smallest are.

How to Find the Absolute Extrema of a Continuous Function f on a Closed Interval

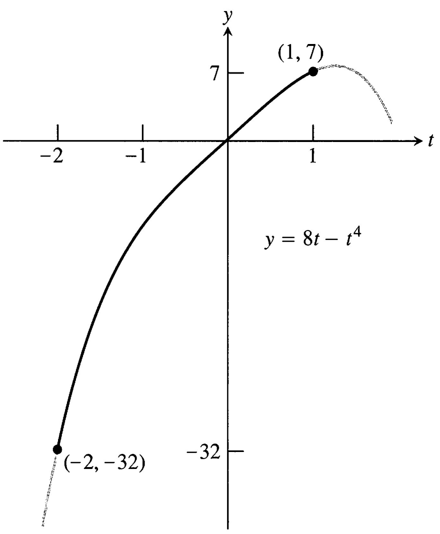
1. Evaluate f at all critical points and endpoints.
2. Take the largest and smallest of these values.

EXAMPLE 3 Find the absolute maximum and minimum values of $f(x) = x^2$ on $[-2, 1]$.

Solution The function is differentiable over its entire domain, so the only critical point is where $f'(x) = 2x = 0$, namely $x = 0$. We need to check the function's values at $x = 0$ and at the endpoints $x = -2$ and $x = 1$:

Critical point value:	$f(0) = 0$
Endpoint values:	$f(-2) = 4$
	$f(1) = 1$

The function has an absolute maximum value of 4 at $x = -2$ and an absolute minimum value of 0 at $x = 0$. ❑



3.8 The extreme values of $g(t) = 8t - t^4$ on $[-2, 1]$ (Example 4).

EXAMPLE 4 Find the absolute extrema values of $g(t) = 8t - t^4$ on $[-2, 1]$.

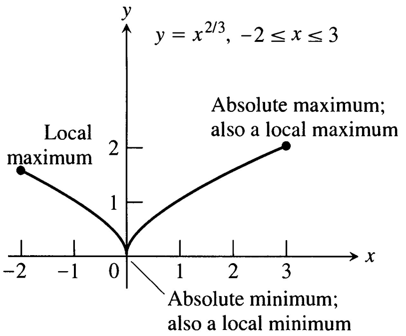
Solution The function is differentiable on its entire domain, so the only critical points occur where $g'(t) = 0$. Solving this equation gives

$$8 - 4t^3 = 0$$
$$t^3 = 2$$
$$t = 2^{1/3},$$

a point not in the given domain. The function's local extrema therefore occur at the endpoints, where we find

$$g(-2) = -32 \qquad \text{(Absolute minimum)}$$
$$g(1) = 7. \qquad \text{(Absolute maximum)}$$

See Fig. 3.8. ❑



3.9 The extreme values of $h(x) = x^{2/3}$ on $[-2, 3]$ occur at $x = 0$ and $x = 3$ (Example 5).

EXAMPLE 5 Find the absolute extrema of $h(x) = x^{2/3}$ on $[-2, 3]$.

Solution The first derivative

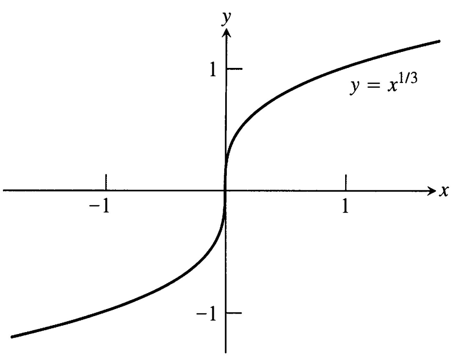
$$h'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}$$

has no zeros but is undefined at $x = 0$. The values of h at this one critical point and at the endpoints $x = -2$ and $x = 3$ are

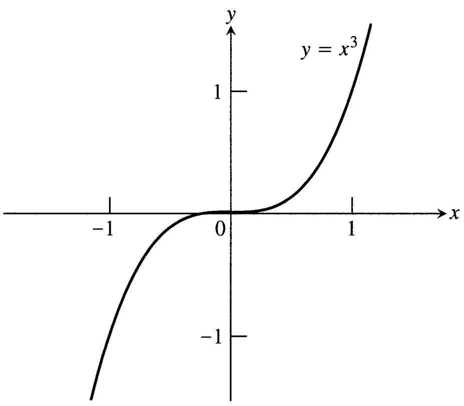
$$h(0) = 0$$
$$h(-2) = (-2)^{2/3} = 4^{1/3}$$
$$h(3) = (3)^{2/3} = 9^{1/3}.$$

The absolute maximum value is $9^{1/3}$, assumed at $x = 3$; the absolute minimum is 0, assumed at $x = 0$ (Fig. 3.9). ❑

While a function's extrema can occur only at critical points and endpoints, not every critical point or endpoint signals the presence of an extreme value. Figures 3.10 and 3.11 illustrate this for interior points, and Exercise 34 asks you for a function that fails to assume an extreme value at an endpoint of its domain.



3.10 $f(x) = x^{1/3}$ has no extremum at $x = 0$, even though $f'(x) = (1/3)x^{-2/3}$ is undefined at $x = 0$.



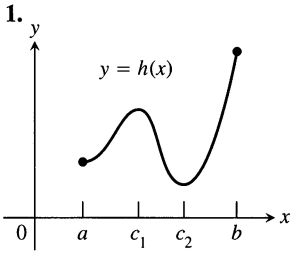
3.11 $g(x) = x^3$ has no extremum at $x = 0$ even though $g'(x) = 3x^2$ is zero at $x = 0$.

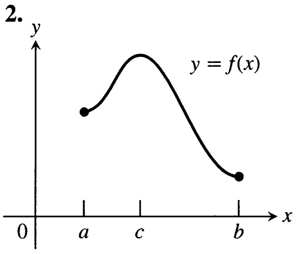
As we will see in Section 3.3, we can determine the behavior of a function f at a critical point c by further examining f', but we must look beyond what f' does at c itself.

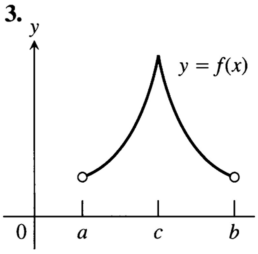
Exercises 3.1

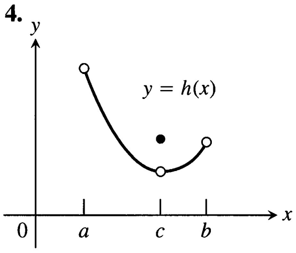
Finding Extrema from Graphs

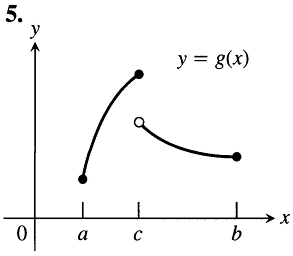
In Exercises 1–6, determine from the graph whether the function has any absolute extreme values on $[a, b]$. Then explain how your answer is consistent with Theorem 1.

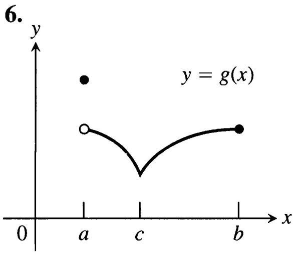
1.


2.


3.


4.


5.


6.


Absolute Extrema on Closed Intervals

In Exercises 7–22, find the absolute maximum and minimum values of each function on the given interval. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

7. $f(x) = \frac{2}{3}x - 5, \quad -2 \le x \le 3$

8. $f(x) = -x - 4, \quad -4 \le x \le 1$

9. $f(x) = x^2 - 1, \quad -1 \le x \le 2$

10. $f(x) = 4 - x^2, \quad -3 \le x \le 1$

11. $F(x) = -\frac{1}{x^2}, \quad 0.5 \le x \le 2$

12. $F(x) = -\frac{1}{x}, \quad -2 \le x \le -1$

13. $h(x) = \sqrt[3]{x}, \quad -1 \le x \le 8$

14. $h(x) = -3x^{2/3}, \quad -1 \le x \le 1$

15. $g(x) = \sqrt{4 - x^2}, \quad -2 \le x \le 1$

16. $g(x) = -\sqrt{5 - x^2}, \quad -\sqrt{5} \le x \le 0$

17. $f(\theta) = \sin\theta, \quad -\frac{\pi}{2} \le \theta \le \frac{5\pi}{6}$

18. $f(\theta) = \tan\theta, \quad -\frac{\pi}{3} \le \theta \le \frac{\pi}{4}$

19. $g(x) = \csc x, \quad \frac{\pi}{3} \le x \le \frac{2\pi}{3}$

20. $g(x) = \sec x, \quad -\frac{\pi}{3} \le x \le \frac{\pi}{6}$

21. $f(t) = 2 - |t|, \quad -1 \le t \le 3$

22. $f(t) = |t - 5|, \quad 4 \le t \le 7$

In Exercises 23–26, find the function's absolute maximum and minimum values and say where they are assumed.

23. $f(x) = x^{4/3}, \quad -1 \le x \le 8$

24. $f(x) = x^{5/3}, \quad -1 \le x \le 8$

25. $g(\theta) = \theta^{3/5}, \quad -32 \le \theta \le 1$

26. $h(\theta) = 3\theta^{2/3}, \quad -27 \le \theta \le 8$

Local Extrema in the Domain

In Exercises 27 and 28, find the values of any local maxima and minima the functions may have on the given domains, and say where they are assumed. Which extrema, if any, are absolute for the given domain?

27. **a)** $f(x) = x^2 - 4, \quad -2 \le x \le 2$
b) $g(x) = x^2 - 4, \quad -2 \le x < 2$
c) $h(x) = x^2 - 4, \quad -2 < x < 2$
d) $k(x) = x^2 - 4, \quad -2 \le x < \infty$
e) $l(x) = x^2 - 4, \quad 0 < x < \infty$

28. a) $f(x) = 2 - 2x^2, \quad -1 \le x \le 1$
b) $g(x) = 2 - 2x^2, \quad -1 < x \le 1$
c) $h(x) = 2 - 2x^2, \quad -1 < x < 1$
d) $k(x) = 2 - 2x^2, \quad -\infty < x \le 1$
e) $l(x) = 2 - 2x^2, \quad -\infty < x < 0$

Theory and Examples

29. The function $f(x) = |x|$ has an absolute minimum value at $x = 0$ even though f is not differentiable at $x = 0$. Is this consistent with Theorem 2? Give reasons for your answer.

30. Why can't the conclusion of Theorem 2 be expected to hold if c is an endpoint of the function's domain?

31. If an even function $f(x)$ has a local maximum value at $x = c$, can anything be said about the value of f at $x = -c$? Give reasons for your answer.

32. If an odd function $g(x)$ has a local minimum value at $x = c$, can anything be said about the value of g at $x = -c$? Give reasons for your answer.

33. We know how to find the extreme values of a continuous function $f(x)$ by investigating its values at critical points and endpoints. But what if there *are* no critical points or endpoints? What happens then? Do such functions really exist? Give reasons for your answers.

34. Give an example of a function defined on $[0, 1]$ that has neither a local maximum nor a local minimum value at 0.

CAS Explorations and Projects

In Exercises 35–40, you will use a CAS to help find the absolute extrema of the given function over the specified closed interval. Perform the following steps:

a) Plot the function over the interval to see general behavior there.
b) Find the interior points where $f' = 0$. (In some exercises you may have to use the numerical equation solver to approximate a solution.) You may want to plot f' as well.
c) Find the interior points where f' does not exist.
d) Evaluate the function at all points found in parts (b) and (c) and at the endpoints of the interval.
e) Find the function's absolute extreme values on the interval and identify where they occur.

35. $f(x) = x^4 - 8x^2 + 4x + 2, \quad \left[-\frac{20}{25}, \frac{64}{25}\right]$

36. $f(x) = -x^4 + 4x^3 - 4x + 1, \quad \left[-\frac{3}{4}, 3\right]$

37. $f(x) = x^{2/3}(3 - x), \quad [-2, 2]$

38. $f(x) = 2 + 2x - 3x^{2/3}, \quad \left[-1, \frac{10}{3}\right]$

39. $f(x) = \sqrt{x} + \cos x, \quad [0, 2\pi]$

40. $f(x) = x^{3/4} - \sin x + \frac{1}{2}, \quad [0, 2\pi]$

3.2 The Mean Value Theorem

When the French mathematician Michel Rolle published his theorem in 1691, his goal was to show that between every two zeros of a polynomial function there always lies a zero of the polynomial we now know to be the function's derivative. (The modern version of the theorem is not restricted to polynomials.)

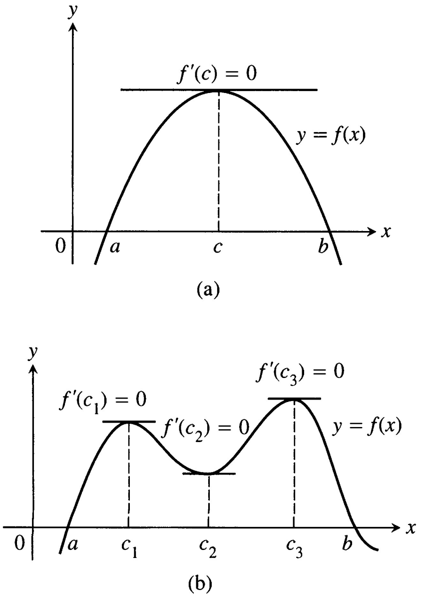
Rolle distrusted the new methods of calculus, however, and spent a great deal of time and energy denouncing their use and attacking l'Hôpital's all too popular (he felt) calculus book. It is ironic that Rolle is known today only for his inadvertent contribution to a field he tried to suppress.

If a body falls freely from rest near the surface of the earth, its position t seconds into the fall is $s = 4.9t^2$ m. From this we deduce that the body's velocity and acceleration are $v = ds/dt = 9.8t$ m/sec and $a = d^2s/dt^2 = 9.8$ m/sec^2. But suppose we started with the body's acceleration. Could we work backward to find its velocity and displacement functions?

What we are really asking here is what functions can have a given derivative. More generally, we might ask what kind of function can have a particular *kind* of derivative. What kind of function has a positive derivative, for instance, or a negative derivative, or a derivative that is always zero? We answer these questions by applying corollaries of the Mean Value Theorem.

Rolle's Theorem

There is strong geometric evidence that between any two points where a differentiable curve crosses the x-axis there is a point on the curve where the tangent is horizontal. A 300-year-old theorem of Michel Rolle (1652–1719) assures us that this is indeed the case.



3.12 Rolle's theorem says that a differentiable curve has at least one horizontal tangent between any two points where it crosses the x-axis. It may have just one (a), or it may have more (b).

Theorem 3
Rolle's Theorem

Suppose that $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If

$$f(a) = f(b) = 0,$$

then there is at least one number c in (a, b) at which

$$f'(c) = 0.$$

See Fig. 3.12.

Proof Being continuous, f assumes absolute maximum and minimum values on $[a, b]$. These can occur only

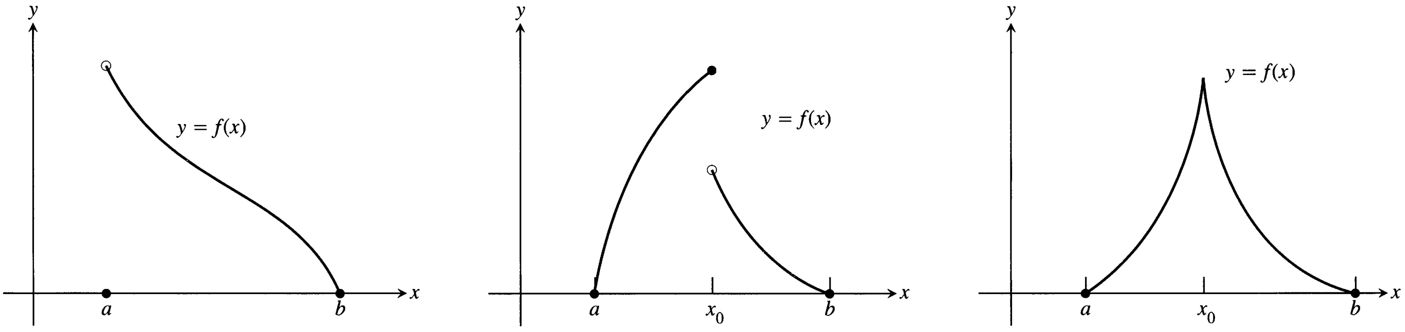
1. at interior points where f' is zero,
2. at interior points where f' does not exist,
3. at the endpoints of the function's domain, in this case a and b.

By hypothesis, f has a derivative at every interior point. That rules out (2), leaving us with interior points where $f' = 0$ and with the two endpoints a and b.

If either the maximum or the minimum occurs at a point c inside the interval, then $f'(c) = 0$ by Theorem 2 in Section 3.1, and we have found a point for Rolle's theorem.

If both maximum and minimum are at a or b, then f is constant, $f' = 0$, and c can be taken anywhere in the interval. This completes the proof. ❑

The hypotheses of Theorem 3 are essential. If they fail at even one point, the graph may not have a horizontal tangent (Fig. 3.13).



(a) Discontinuous at an endpoint

(b) Discontinuous at an interior point

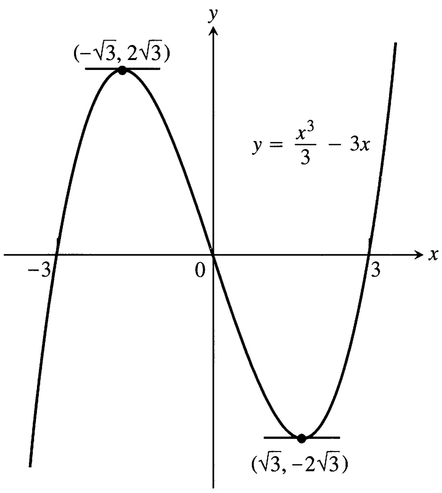
(c) Continuous on $[a, b]$ but not differentiable at some interior point

3.13 No horizontal tangent.

EXAMPLE 1 The polynomial function

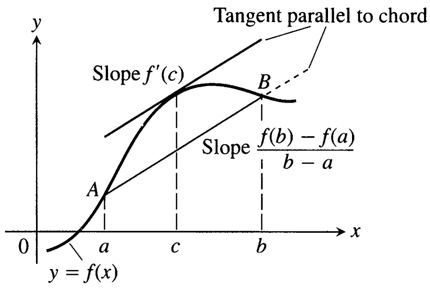
$$f(x) = \frac{x^3}{3} - 3x$$

graphed in Fig. 3.14 (on the following page) is continuous at every point of $[-3, 3]$ and is differentiable at every point of $(-3, 3)$. Since $f(-3) = f(3) = 0$, Rolle's



3.14 As predicted by Rolle's theorem, this curve has horizontal tangents between the points where it crosses the *x*-axis (Example 1).

theorem says that f' must be zero at least once in the open interval between $a = -3$ and $b = 3$. In fact, $f'(x) = x^2 - 3$ is zero twice in this interval, once at $x = -\sqrt{3}$ and again at $x = \sqrt{3}$. ❑



3.15 Geometrically, the Mean Value Theorem says that somewhere between *A* and *B* the curve has at least one tangent parallel to chord *AB*.

The Mean Value Theorem

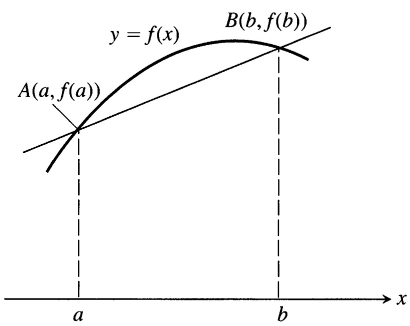
The Mean Value Theorem is a slanted version of Rolle's theorem (Fig. 3.15). There is a point where the tangent is parallel to chord *AB*.

Theorem 4
The Mean Value Theorem

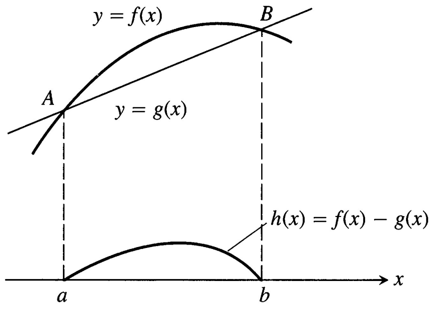
Suppose $y = f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one point c in (a, b) at which

$$\frac{f(b) - f(a)}{b - a} = f'(c). \qquad (1)$$

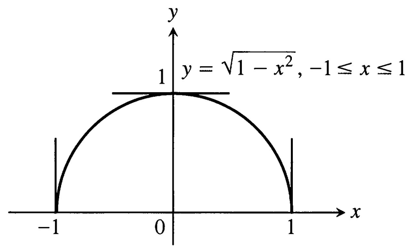
Proof We picture the graph of f as a curve in the plane and draw a line through the points $A(a, f(a))$ and $B(b, f(b))$ (see Fig. 3.16). The line is the graph of the



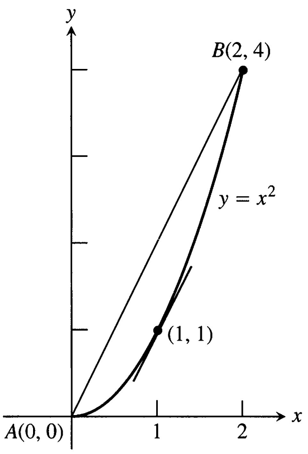
3.16 The graph of *f* and the chord *AB* over the interval $[a, b]$.



3.17 The chord AB in Fig. 3.16 is the graph of the function $g(x)$. The function $h(x) = f(x) - g(x)$ gives the vertical distance between the graphs of f and g at x.



3.18 The function $f(x) = \sqrt{1 - x^2}$ satisfies the hypotheses (and conclusion) of the Mean Value Theorem on $[-1, 1]$ even though f is not differentiable at -1 and 1.



3.19 As we find in Example 2, $c = 1$ is where the tangent is parallel to the chord.

function

$$g(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \tag{2}$$

(point–slope equation). The vertical difference between the graphs of f and g at x is

$$\begin{aligned} h(x) &= f(x) - g(x) \\ &= f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a). \end{aligned} \tag{3}$$

Figure 3.17 shows the graphs of f, g, and h together.

The function h satisfies the hypotheses of Rolle's theorem on $[a, b]$. It is continuous on $[a, b]$ and differentiable on (a, b) because both f and g are. Also, $h(a) = h(b) = 0$ because the graphs of f and g both pass through A and B. Therefore, $h' = 0$ at some point c in (a, b). This is the point we want for Eq. (1).

To verify Eq. (1), we differentiate both sides of Eq. (3) with respect to x and then set $x = c$:

$$h'(x) = f'(x) - \frac{f(b) - f(a)}{b - a} \qquad \text{Derivative of Eq. (3) . . .}$$

$$h'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} \qquad \text{. . . with } x = c$$

$$0 = f'(c) - \frac{f(b) - f(a)}{b - a} \qquad h'(c) = 0$$

$$f'(c) = \frac{f(b) - f(a)}{b - a}, \qquad \text{Rearranged}$$

which is what we set out to prove. ❑

Notice that the hypotheses of the Mean Value Theorem do not require f to be differentiable at either a or b. Continuity at a and b is enough (Fig. 3.18).

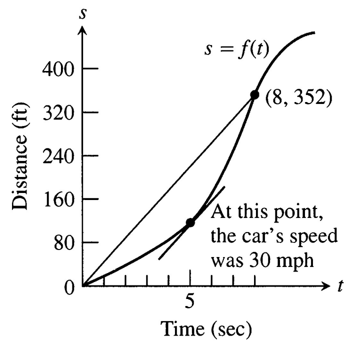
We usually do not know any more about the number c than the theorem tells, which is that c exists. In a few cases we can satisfy our curiosity about the identity of c, as in the next example. However, our ability to identify c is the exception rather than the rule, and the importance of the theorem lies elsewhere.

EXAMPLE 2 The function $f(x) = x^2$ (Fig. 3.19) is continuous for $0 \le x \le 2$ and differentiable for $0 < x < 2$. Since $f(0) = 0$ and $f(2) = 4$, the Mean Value Theorem says that at some point c in the interval, the derivative $f'(x) = 2x$ must have the value $(4 - 0)/(2 - 0) = 2$. In this (exceptional) case we can identify c by solving the equation $2c = 2$ to get $c = 1$. ❑

Physical Interpretations

If we think of the number $(f(b) - f(a))/(b - a)$ as the average change in f over $[a, b]$ and $f'(c)$ as an instantaneous change, then the Mean Value Theorem says that at some interior point the instantaneous change must equal the average change over the entire interval.

EXAMPLE 3 If a car accelerating from zero takes 8 sec to go 352 ft, its average velocity for the 8-sec interval is $352/8 = 44$ ft/sec. At some point during the acceleration, the Mean Value Theorem says, the speedometer must read exactly 30 mph (44 ft/sec) (Fig. 3.20). ❑



3.20 Distance vs. elapsed time for the car in Example 3.

Corollaries and Some Answers

At the beginning of the section, we asked what kind of function has a zero derivative. The first corollary of the Mean Value Theorem provides the answer.

Corollary 1
Functions with Zero Derivatives Are Constant
If $f'(x) = 0$ at each point of an interval I, then $f(x) = C$ for all x in I, where C is a constant.

We know that if a function f has a constant value on an interval I, then f is differentiable on I and $f'(x) = 0$ for all x in I. Corollary 1 provides the converse.

Proof of Corollary 1 We want to show that f has a constant value on I. We do so by showing that if x_1 and x_2 are any two points in I, then $f(x_1) = f(x_2)$.

Suppose that x_1 and x_2 are two points in I, numbered from left to right so that $x_1 < x_2$. Then f satisfies the hypotheses of the Mean Value Theorem on $[x_1, x_2]$: It is differentiable at every point of $[x_1, x_2]$, and hence continuous at every point as well. Therefore,

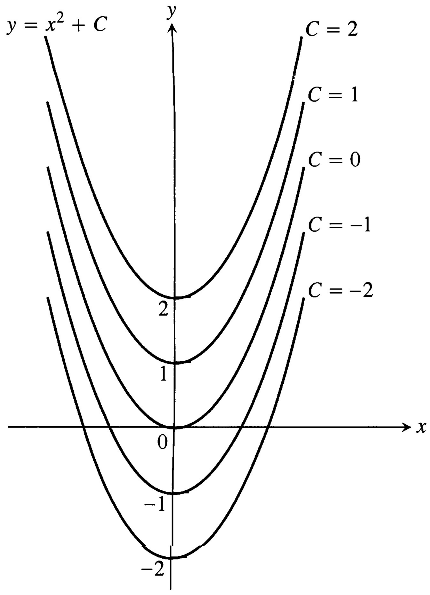
$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$$

at some point c between x_1 and x_2. Since $f' = 0$ throughout I, this equation translates successively into

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0, \quad f(x_2) - f(x_1) = 0, \quad \text{and} \quad f(x_1) = f(x_2).$$

❑

At the beginning of the section, we also asked if we could work backward from the acceleration of a body falling freely from rest to find the body's velocity and displacement functions. The answer is yes, and it is a consequence of the next corollary.



3.21 From a geometric point of view, Corollary 2 of the Mean Value Theorem says that the graphs of functions with identical derivatives can differ only by a vertical shift. The graphs of the functions with derivative $2x$ are the parabolas $y = x^2 + C$, shown here for selected values of C.

Corollary 2
Functions with the Same Derivative Differ by a Constant

If $f'(x) = g'(x)$ at each point of an interval I, then there exists a constant C such that $f(x) = g(x) + C$ for all x in I.

Proof At each point x in I the derivative of the difference function $h = f - g$ is

$$h'(x) = f'(x) - g'(x) = 0.$$

Thus, $h(x) = C$ on I (Corollary 1). That is, $f(x) - g(x) = C$ on I, so $f(x) = g(x) + C$. ❑

Corollary 2 says that functions can have identical derivatives on an interval only if their values on the interval have a constant difference. We know, for instance, that the derivative of $f(x) = x^2$ on $(-\infty, \infty)$ is $2x$. Any other function with derivative $2x$ on $(-\infty, \infty)$ must have the formula $x^2 + C$ for some value of C (Fig. 3.21).

EXAMPLE 4 Find the function $f(x)$ whose derivative is $\sin x$ and whose graph passes through the point (0, 2).

Solution Since $f(x)$ has the same derivative as $g(x) = -\cos x$, we know that $f(x) = -\cos x + C$ for some constant C. The value of C can be determined from the condition that $f(0) = 2$ (the graph of f passes through (0, 2)):

$$f(0) = -\cos(0) + C = 2, \qquad \text{so} \qquad C = 3.$$

The formula for f is $f(x) = -\cos x + 3$. ❑

Finding Velocity and Position from Acceleration

Here is how to find the velocity and displacement functions of a body falling freely from rest with acceleration 9.8 m/sec^2.

We know that $v(t)$ is some function whose derivative is 9.8. We also know that the derivative of $g(t) = 9.8t$ is 9.8. By Corollary 2,

$$v(t) = 9.8t + C \tag{4}$$

for some constant C. Since the body falls from rest, $v(0) = 0$. Thus

$$9.8(0) + C = 0, \qquad \text{and} \qquad C = 0.$$

The velocity function must be $v(t) = 9.8t$. How about the position function $s(t)$?

We know that $s(t)$ is some function whose derivative is $9.8t$. We also know that the derivative of $h(t) = 4.9t^2$ is $9.8t$. By Corollary 2,

$$s(t) = 4.9t^2 + C \tag{5}$$

for some constant C. Since $s(0) = 0$,

$$4.9(0)^2 + C = 0, \qquad \text{and} \qquad C = 0.$$

The position function must be $s(t) = 4.9t^2$.

The ability to find functions from their rates of change is one of the great powers we gain from calculus. As we will see, it lies at the heart of the mathematical developments in Chapter 4. We will continue the story there.

Increasing Functions and Decreasing Functions

At the beginning of the section we asked what kinds of functions have positive derivatives or negative derivatives. The answer, provided by the Mean Value Theorem's third corollary, is this: The only functions with positive derivatives are increasing functions; the only functions with negative derivatives are decreasing functions.

Definitions

Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

1. f **increases** on I if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.
2. f **decreases** on I if $x_1 < x_2 \Rightarrow f(x_2) < f(x_1)$.

Corollary 3
The First Derivative Test for Increasing and Decreasing

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).

If $f' > 0$ at each point of (a, b), then f increases on $[a, b]$.

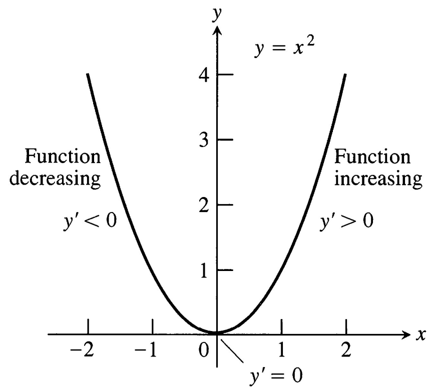
If $f' < 0$ at each point of (a, b), then f decreases on $[a, b]$.

Proof Let x_1 and x_2 be two points in $[a, b]$ with $x_1 < x_2$. The Mean Value Theorem applied to f on $[x_1, x_2]$ says that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1) \tag{6}$$

for some c between x_1 and x_2. The sign of the right-hand side of Eq. (6) is the same as the sign of $f'(c)$ because $x_2 - x_1$ is positive. Therefore, $f(x_2) > f(x_1)$ if f' is positive on (a, b), and $f(x_2) < f(x_1)$ if f' is negative on (a, b). ❑

EXAMPLE 5 The function $f(x) = x^2$ decreases on $(-\infty, 0)$, where $f'(x) = 2x < 0$. It increases on $(0, \infty)$, where $f'(x) = 2x > 0$ (Fig. 3.22). ❑



3.22 The graph for Example 5.

Exercises 3.2

Finding c in the Mean Value Theorem

Find the value or values of c that satisfy the equation

$$\frac{f(b)-f(a)}{b-a}=f'(c)$$

in the conclusion of the Mean Value Theorem for the functions and intervals in Exercises 1–4.

1. $f(x)=x^2+2x-1,\quad [0,1]$

2. $f(x)=x^{2/3},\quad [0,1]$

3. $f(x)=x+\frac{1}{x},\quad \left[\frac{1}{2},2\right]$

4. $f(x)=\sqrt{x-1},\quad [1,3]$

Checking and Using Hypotheses

Which of the functions in Exercises 5–8 satisfy the hypotheses of the Mean Value Theorem on the given interval, and which do not? Give reasons for your answers.

5. $f(x)=x^{2/3},\quad [-1,8]$

6. $f(x)=x^{4/5},\quad [0,1]$

7. $f(x)=\sqrt{x(1-x)},\quad [0,1]$

8. $f(x)=\begin{cases}\dfrac{\sin x}{x}, & -\pi\le x<0\\ 0, & x=0\end{cases}$

9. The function

$$f(x)=\begin{cases}x, & 0\le x<1\\ 0, & x=1\end{cases}$$

is zero at $x=0$ and $x=1$ and differentiable on (0, 1), but its derivative on (0, 1) is never zero. How can this be? Doesn't Rolle's theorem say the derivative has to be zero somewhere in (0, 1)? Give reasons for your answer.

10. For what values of a, m, and b does the function

$$f(x)=\begin{cases}3, & x=0\\ -x^2+3x+a, & 0<x<1\\ mx+b, & 1\le x\le 2\end{cases}$$

satisfy the hypotheses of the Mean Value Theorem on the interval [0, 2]?

Roots (Zeros)

11. a) Plot the zeros of each polynomial on a line together with the zeros of its first derivative.

i) $y=x^2-4$
ii) $y=x^2+8x+15$
iii) $y=x^3-3x^2+4=(x+1)(x-2)^2$
iv) $y=x^3-33x^2+216x=x(x-9)(x-24)$

b) Use Rolle's theorem to prove that between every two zeros of $x^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ there lies a zero of

$$nx^{n-1}+(n-1)a_{n-1}x^{n-2}+\cdots+a_1.$$

12. Suppose that f'' is continuous on $[a, b]$ and that f has three zeros in the interval. Show that f'' has at least one zero in (a, b). Generalize this result.

13. Show that if $f''>0$ throughout an interval $[a, b]$, then f' has at most one zero in $[a, b]$. What if $f''<0$ throughout $[a, b]$ instead?

14. Show that a cubic polynomial can have at most three real zeros.

Theory and Examples

15. Show that at some instant during a 2-h automobile trip the car's speedometer reading will equal the average speed for the trip.

16. *Temperature change.* It took 14 sec for a thermometer to rise from −19°C to 100°C when it was taken from a freezer and placed in boiling water. Show that somewhere along the way the mercury was rising at exactly 8.5°C/sec.

17. Suppose that f is differentiable on [0, 1] and that its derivative is never zero. Show that $f(0)\ne f(1)$.

18. Show that $|\sin b-\sin a|\le|b-a|$ for any numbers a and b.

19. Suppose that f is differentiable on $[a, b]$ and that $f(b)<f(a)$. Can you then say anything about the values of f' on $[a, b]$?

20. Suppose that f and g are differentiable on $[a, b]$ and that $f(a)=g(a)$ and $f(b)=g(b)$. Show that there is at least one point between a and b where the tangents to the graphs of f and g are parallel.

21. Let f be differentiable at every value of x and suppose that $f(1)=1$, that $f'<0$ on $(-\infty, 1)$, and that $f'>0$ on $(1, \infty)$.

a) Show that $f(x)\ge 1$ for all x.
b) Must $f'(1)=0$? Explain.

22. Let $f(x)=px^2+qx+r$ be a quadratic function defined on a closed interval $[a, b]$. Show that there is exactly one point c in (a, b) at which f satisfies the conclusion of the Mean Value Theorem.

23. *A surprising graph.* Graph the function

$$f(x)=\sin x\sin(x+2)-\sin^2(x+1).$$

What does the graph do? Why does the function behave this way? Give reasons for your answers.

24. If the graphs of two functions $f(x)$ and $g(x)$ start at the same point in the plane and the functions have the same rate of change at every point, do the graphs have to be identical? Give reasons for your answer.

25. a) Show that $g(x)=1/x$ decreases on every interval in its domain.

b) If the conclusion in (a) is really true, how do you explain the fact that $g(1) = 1$ is actually greater than $g(-1) = -1$?

26. Let f be a function defined on an interval $[a, b]$. What conditions could you place on f to guarantee that

$$\min f' \le \frac{f(b) - f(a)}{b - a} \le \max f',$$

where $\min f'$ and $\max f'$ refer to the minimum and maximum values of f' on $[a, b]$? Give reasons for your answer.

27. CALCULATOR Use the inequalities in Exercise 26 to estimate $f(0.1)$ if $f'(x) = 1/(1 + x^4 \cos x)$ for $0 \le x \le 0.1$ and $f(0) = 1$.

28. CALCULATOR Use the inequalities in Exercise 26 to estimate $f(0.1)$ if $f'(x) = 1/(1 - x^4)$ for $0 \le x \le 0.1$ and $f(0) = 2$.

29. *The geometric mean of a and b.* The **geometric mean** of two positive numbers a and b is the number $\sqrt{ab}$. Show that the value of c in the conclusion of the Mean Value Theorem for $f(x) = 1/x$ on an interval $[a, b]$ of positive numbers is $c = \sqrt{ab}$.

30. *The arithmetic mean of a and b.* The **arithmetic mean** of two numbers a and b is the number $(a + b)/2$. Show that the value of c in the conclusion of the Mean Value Theorem for $f(x) = x^2$ on any interval $[a, b]$ is $c = (a + b)/2$.

Finding Functions from Derivatives

31. Suppose that $f(-1) = 3$ and that $f'(x) = 0$ for all x. Must $f(x) = 3$ for all x? Give reasons for your answer.

32. Suppose that $f(0) = 5$ and that $f'(x) = 2$ for all x. Must $f(x) = 2x + 5$ for all x? Give reasons for your answer.

33. Suppose that $f'(x) = 2x$ for all x. Find $f(2)$ if

a) $f(0) = 0$ **b)** $f(1) = 0$ **c)** $f(-2) = 3$.

34. What can be said about functions whose derivatives are constant? Give reasons for your answer.

In Exercises 35–40, find all possible functions with the given derivative.

35. a) $y' = x$ **b)** $y' = x^2$ **c)** $y' = x^3$

36. a) $y' = 2x$
b) $y' = 2x - 1$
c) $y' = 3x^2 + 2x - 1$

37. a) $y' = -\dfrac{1}{x^2}$

b) $y' = 1 - \dfrac{1}{x^2}$

c) $y' = 5 + \dfrac{1}{x^2}$

38. a) $y' = \dfrac{1}{2\sqrt{x}}$

b) $y' = \dfrac{1}{\sqrt{x}}$

c) $y' = 4x - \dfrac{1}{\sqrt{x}}$

39. a) $y' = \sin 2t$ **b)** $y' = \cos \dfrac{t}{2}$

c) $y' = \sin 2t + \cos \dfrac{t}{2}$

40. a) $y' = \sec^2 \theta$ **b)** $y' = \sqrt{\theta}$

c) $y' = \sqrt{\theta} - \sec^2 \theta$

In Exercises 41–44, find the function with the given derivative whose graph passes through the point P.

41. $f'(x) = 2x - 1, \quad P(0, 0)$

42. $g'(x) = \dfrac{1}{x^2} + 2x, \quad P(-1, 1)$

43. $r'(\theta) = 8 - \csc^2 \theta, \quad P\left(\dfrac{\pi}{4}, 0\right)$

44. $r'(t) = \sec t \tan t - 1, \quad P(0, 0)$

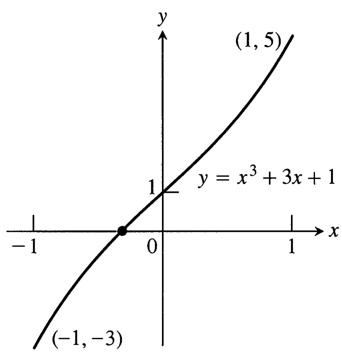
Counting Zeros

When we solve an equation $f(x) = 0$ numerically, we usually want to know beforehand how many solutions to look for in a given interval. With the help of Corollary 3 we can sometimes find out.

Suppose that

1. f is continuous on $[a, b]$ and differentiable on (a, b),
2. $f(a)$ and $f(b)$ have opposite signs,
3. $f' > 0$ on (a, b) or $f' < 0$ on (a, b).

Then f has exactly one zero between a and b: It cannot have more than one because it is either increasing on $[a, b]$ or decreasing on $[a, b]$. Yet it has at least one, by the Intermediate Value Theorem (Section 1.5). For example, $f(x) = x^3 + 3x + 1$ has exactly one zero on $[-1, 1]$ because f is differentiable on $[-1, 1]$, $f(-1) = -3$ and $f(1) = 5$ have opposite signs, and $f'(x) = 3x^2 + 3 > 0$ for all x (Fig. 3.23).



3.23 The only real zero of the polynomial $y = x^3 + 3x + 1$ is the one shown here between -1 and 0.

Show that the functions in Exercises 45–52 have exactly one zero in the given interval.

45. $f(x) = x^4 + 3x + 1, \quad [-2, -1]$

46. $f(x) = x^3 + \dfrac{4}{x^2} + 7, \quad (-\infty, 0)$

47. $g(t) = \sqrt{t} + \sqrt{1+t} - 4, \quad (0, \infty)$

48. $g(t) = \dfrac{1}{1-t} + \sqrt{1+t} - 3.1, \quad (-1, 1)$

49. $r(\theta) = \theta + \sin^2\left(\dfrac{\theta}{3}\right) - 8, \quad (-\infty, \infty)$

50. $r(\theta) = 2\theta - \cos^2\theta + \sqrt{2}, \quad (-\infty, \infty)$

51. $r(\theta) = \sec\theta - \dfrac{1}{\theta^3} + 5, \quad (0, \pi/2)$

52. $r(\theta) = \tan\theta - \cot\theta - \theta, \quad (0, \pi/2)$

CAS Exploration

53. *Rolle's original theorem*

a) Construct a polynomial $f(x)$ that has zeros at $x = -2, -1, 0, 1,$ and 2.

b) Graph f and its derivative f' together. How is what you see related to Rolle's original theorem? (See the marginal note on Rolle.)

c) Do $g(x) = \sin x$ and its derivative g' illustrate the same phenomenon?

d) How would you state and prove Rolle's original theorem in light of what we know today?

3.3 The First Derivative Test for Local Extreme Values

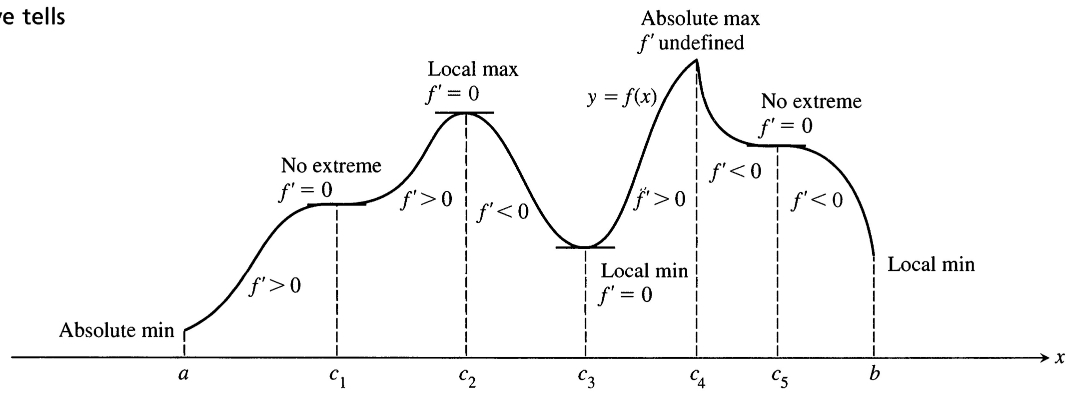
This section shows how to test a function's critical points for the presence of local extreme values.

The Test

As we see once again in Fig. 3.24, a function f may have local extrema at some critical points while failing to have local extrema at others. The key is the sign of f' in the point's immediate vicinity. As x moves from left to right, the values of f increase where $f' > 0$ and decrease where $f' < 0$.

At the points where f has a minimum value, we see that $f' < 0$ on the interval immediately to the left and $f' > 0$ on the interval immediately to the right. (If the point is an endpoint, there is only the interval on the appropriate side to consider.) This means that the curve is falling (values decreasing) on the left of the minimum value and rising (values increasing) on its right. Similarly, at the points where f has a maximum value, $f' > 0$ on the interval immediately to the left and $f' < 0$ on the interval immediately to the right. This means that the curve is rising (values increasing) on the left of the maximum value and falling (values decreasing) on its right.

These observations lead to a test for the presence of local extreme values.



3.24 A function's first derivative tells how the graph rises and falls.

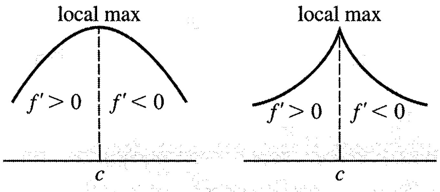
Theorem 5

The First Derivative Test for Local Extreme Values

The following test applies to a continuous function $f(x)$.

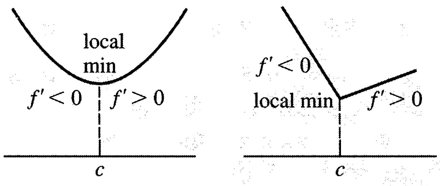
At a critical point c:

1. If f' changes from positive to negative at c ($f' > 0$ for $x < c$ and $f' < 0$ for $x > c$), then f has a local maximum value at c.



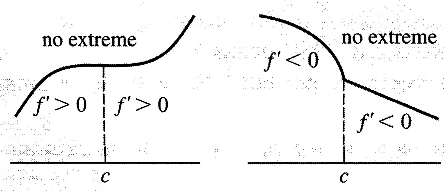
(a) $f'(c) = 0$ (b) $f'(c)$ undefined

2. If f' changes from negative to positive at c ($f' < 0$ for $x < c$ and $f' > 0$ for $x > c$), then f has a local minimum value at c.



(a) $f'(c) = 0$ (b) $f'(c)$ undefined

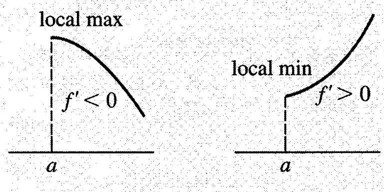
3. If f' does not change sign at c (f' has the same sign on both sides of c), then f has no local extreme value at c.



(a) $f'(c) = 0$ (b) $f'(c)$ undefined

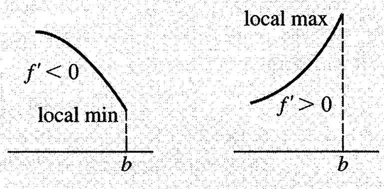
At a left endpoint a:

If $f' < 0$ ($f' > 0$) for $x > a$, then f has a local maximum (minimum) value at a.



At a right endpoint b:

If $f' < 0$ ($f' > 0$) for $x < b$, then f has a local minimum (maximum) value at b.

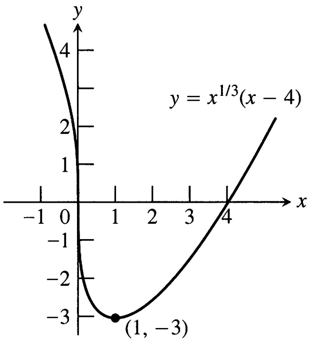


EXAMPLE 1 Find the critical points of

$$f(x) = x^{1/3}(x - 4) = x^{4/3} - 4x^{1/3}.$$

Identify the intervals on which f is increasing and decreasing. Find the function's local and absolute extreme values.

Solution The function f is defined for all real numbers and is continuous (Fig. 3.25).



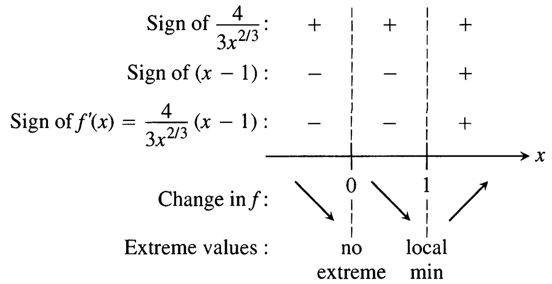
3.25 The graph of $y = x^{1/3}(x - 4)$ (Example 1).

The first derivative

$$f'(x) = \frac{d}{dx}\left(x^{4/3} - 4x^{1/3}\right) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3}$$
$$= \frac{4}{3}x^{-2/3}(x - 1) = \frac{4(x - 1)}{3x^{2/3}}$$

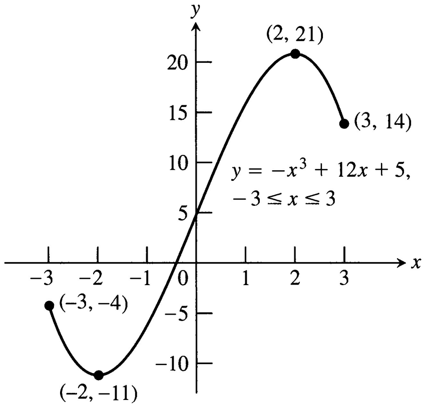
is zero at $x = 1$ and undefined at $x = 0$. There are no endpoints in f's domain, so the critical points, $x = 0$ and $x = 1$, are the only places where f might have an extreme value of any kind.

These critical points divide the x-axis into intervals on which f' is either positive or negative. The sign pattern of f' reveals the behavior of f both between and at the critical points. We can display the information in a picture like the following.



To make the picture, we marked the critical points on the x-axis, noted the sign of each factor of f' on the intervals between the points, and "multiplied" the signs of the factors to find the sign of f'. We then applied Corollary 3 of the Mean Value Theorem to determine that f decreases (↘) on $(-\infty, 0)$, decreases on (0, 1), and increases (↗) on $(1, \infty)$. Theorem 5 tells us that f has no extreme at $x = 0$ (f' does not change sign) and that f has a local minimum at $x = 1$ (f' changes from negative to positive).

The value of the local minimum is $f(1) = 1^{1/3}(1 - 4) = -3$. This is also an absolute minimum because the function's values fall toward it from the left and rise away from it on the right. Figure 3.25 shows this value in relation to the function's graph. ❑



3.26 The graph of $g(x) = -x^3 + 12x + 5$, $-3 \le x \le 3$ (Example 2).

EXAMPLE 2 Find the intervals on which

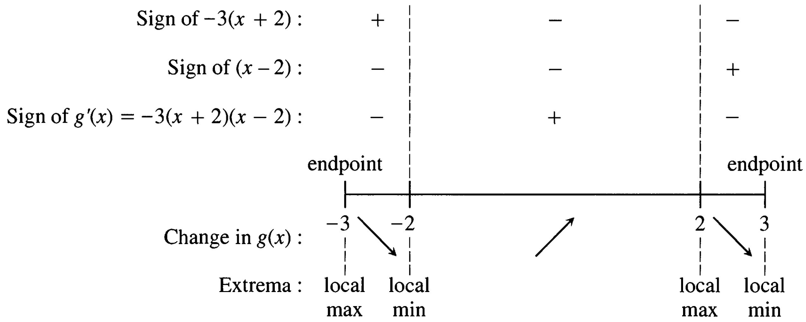
$$g(x) = -x^3 + 12x + 5, \qquad -3 \le x \le 3$$

is increasing and decreasing. Where does the function assume extreme values and what are these values?

Solution The function f is continuous on its domain, $[-3, 3]$ (Fig. 3.26). The first derivative

$$g'(x) = -3x^2 + 12 = -3(x^2 - 4)$$
$$= -3(x + 2)(x - 2),$$

defined at all points of $[-3, 3]$, is zero at $x = -2$ and $x = 2$. These critical points divide the domain of g into intervals on which g' is either positive or negative. We analyze the behavior of g by picturing the sign pattern of g':



We conclude that g has local maxima at $x = -3$ and $x = 2$ and local minima at $x = -2$ and $x = 3$. The corresponding values of $g(x) = -x^3 + 12x + 5$ are

Local maxima: $g(-3) = -4, \quad g(2) = 21$

Local minima: $g(-2) = -11, \quad g(3) = 14.$

Since g is defined on a closed interval, we also know that $g(-2)$ is the absolute minimum and $g(2)$ is the absolute maximum. Figure 3.26 shows these values in relation to the function's graph. ❑

Exercises 3.3

Analyzing f Given f'

Answer the following questions about the functions whose derivatives are given in Exercises 1–8:

a) What are the critical points of f?
b) On what intervals is f increasing or decreasing?
c) At what points, if any, does f assume local maximum and minimum values?

1. $f'(x) = x(x-1)$

2. $f'(x) = (x-1)(x+2)$

3. $f'(x) = (x-1)^2(x+2)$

4. $f'(x) = (x-1)^2(x+2)^2$

5. $f'(x) = (x-1)(x+2)(x-3)$

6. $f'(x) = (x-7)(x+1)(x+5)$

7. $f'(x) = x^{-1/3}(x+2)$

8. $f'(x) = x^{-1/2}(x-3)$

Extremes of Given Functions

In Exercises 9–28:

a) Find the intervals on which the function is increasing and decreasing.
b) Then identify the function's local extreme values, if any, saying where they are taken on.
c) Which, if any, of the extreme values are absolute?
d) GRAPHER You may wish to support your findings with a graphing calculator or computer grapher.

9. $g(t) = -t^2 - 3t + 3$

10. $g(t) = -3t^2 + 9t + 5$

11. $h(x) = -x^3 + 2x^2$

12. $h(x) = 2x^3 - 18x$

13. $f(\theta) = 3\theta^2 - 4\theta^3$

14. $f(\theta) = 6\theta - \theta^3$

15. $f(r) = 3r^3 + 16r$

16. $h(r) = (r+7)^3$

17. $f(x) = x^4 - 8x^2 + 16$

18. $g(x) = x^4 - 4x^3 + 4x^2$

19. $H(t) = \frac{3}{2}t^4 - t^6$

20. $K(t) = 15t^3 - t^5$

21. $g(x) = x\sqrt{8 - x^2}$

22. $g(x) = x^2\sqrt{5 - x}$

23. $f(x) = \frac{x^2 - 3}{x - 2}, \quad x \neq 2$

24. $f(x) = \frac{x^3}{3x^2 + 1}$

25. $f(x) = x^{1/3}(x+8)$

26. $g(x) = x^{2/3}(x+5)$

27. $h(x) = x^{1/3}(x^2 - 4)$

28. $k(x) = x^{2/3}(x^2 - 4)$

Extremes on Half-Open Intervals

In Exercises 29–36:

a) Identify the function's local extreme values in the given domain, and say where they are assumed.
b) Which of the extreme values, if any, are absolute?
c) GRAPHER You may wish to support your findings with a graphing calculator or computer grapher.

29. $f(x) = 2x - x^2, \quad -\infty < x \leq 2$

30. $f(x) = (x+1)^2, \quad -\infty < x \leq 0$

31. $g(x) = x^2 - 4x + 4, \quad 1 \le x < \infty$

32. $g(x) = -x^2 - 6x - 9, \quad -4 \le x < \infty$

33. $f(t) = 12t - t^3, \quad -3 \le t < \infty$

34. $f(t) = t^3 - 3t^2, \quad -\infty < t \le 3$

35. $h(x) = \dfrac{x^3}{3} - 2x^2 + 4x, \quad 0 \le x < \infty$

36. $k(x) = x^3 + 3x^2 + 3x + 1, \quad -\infty < x \le 0$

Graphing Calculator or Computer Grapher

In Exercises 37–40:

a) Find the local extrema of each function on the given interval, and say where they are assumed.

b) GRAPHER Graph the function and its derivative together. Comment on the behavior of f in relation to the signs and values of f'.

37. $f(x) = \dfrac{x}{2} - 2\sin\dfrac{x}{2}, \quad 0 \le x \le 2\pi$

38. $f(x) = -2\cos x - \cos^2 x, \quad -\pi \le x \le \pi$

39. $f(x) = \csc^2 x - 2\cot x, \quad 0 < x < \pi$

40. $f(x) = \sec^2 x - 2\tan x, \quad \dfrac{-\pi}{2} < x < \dfrac{\pi}{2}$

Theory and Examples

Show that the functions in Exercises 41 and 42 have local extreme values at the given values of θ, and say which kind of local extreme the function has.

41. $h(\theta) = 3\cos\dfrac{\theta}{2}, \quad 0 \le \theta \le 2\pi, \quad$ at $\theta = 0$ and $\theta = 2\pi$

42. $h(\theta) = 5\sin\dfrac{\theta}{2}, \quad 0 \le \theta \le \pi, \quad$ at $\theta = 0$ and $\theta = \pi$

43. Sketch the graph of a differentiable function $y = f(x)$ through the point (1, 1) if $f'(1) = 0$ and
   a) $f'(x) > 0$ for $x < 1$ and $f'(x) < 0$ for $x > 1$;
   b) $f'(x) < 0$ for $x < 1$ and $f'(x) > 0$ for $x > 1$;
   c) $f'(x) > 0$ for $x \ne 1$;
   d) $f'(x) < 0$ for $x \ne 1$.

44. Sketch the graph of a differentiable function $y = f(x)$ that has
   a) a local minimum at (1, 1) and a local maximum at (3, 3);
   b) a local maximum at (1, 1) and a local minimum at (3, 3);
   c) local maxima at (1, 1) and (3, 3);
   d) local minima at (1, 1) and (3, 3).

45. Sketch the graph of a continuous function $y = g(x)$ such that
   a) $g(2) = 2$, $0 < g' < 1$ for $x < 2$, $g'(x) \to 1^-$ as $x \to 2^-$, $-1 < g' < 0$ for $x > 2$, and $g'(x) \to -1^+$ as $x \to 2^+$;
   b) $g(2) = 2$, $g' < 0$ for $x < 2$, $g'(x) \to -\infty$ as $x \to 2^-$, $g' > 0$ for $x > 2$, and $g'(x) \to \infty$ as $x \to 2^+$.

46. Sketch the graph of a continuous function $y = h(x)$ such that
   a) $h(0) = 0$, $-2 \le h(x) \le 2$ for all x, $h'(x) \to \infty$ as $x \to 0^-$, and $h'(x) \to -\infty$ as $x \to 0^+$;
   b) $h(0) = 0$, $-2 \le h(x) \le 0$ for all x, $h'(x) \to \infty$ as $x \to 0^-$, and $h'(x) \to -\infty$ as $x \to 0^+$.

47. As x moves from left to right through the point $c = 2$, is the graph of $f(x) = x^3 - 3x + 2$ rising, or is it falling? Give reasons for your answer.

48. Find the intervals on which the function $f(x) = ax^2 + bx + c$, $a \ne 0$, is increasing and decreasing. Describe the reasoning behind your answer.

3.4 Graphing with y' and y''

In Section 3.1, we saw the role played by the first derivative in locating a function's extreme values. A function can have extreme values only at the endpoints of its domain and at its critical points. We also saw that critical points do not necessarily yield extreme values. In Section 3.2, we saw that almost all the information about a differentiable function is contained in its derivative. To recover the function completely, the only additional information we need is the value of the function at any one single point. If a function's derivative is $2x$ and the graph passes through the origin, the function must be x^2. If a function's derivative is $2x$ and the graph passes through the point (0, 4), the function must be $x^2 + 4$.

In Section 3.3, we extended our ability to recover information from a function's first derivative by showing how to use it to tell exactly what happens at a critical point. We can tell whether there really is an extreme value there or whether the graph just continues to rise or fall.

In the present section, we show how to determine the way the graph of a