56. Give an example of functions f and g, both continuous at $x = 0$, for which the composite $f \circ g$ is discontinuous at $x = 0$. Does this contradict Theorem 8? Give reasons for your answer.

57. Is it true that a continuous function that is never zero on an interval never changes sign on that interval? Give reasons for your answer.

58. Is it true that if you stretch a rubber band by moving one end to the right and the other to the left, some point of the band will end up in its original position? Give reasons for your answer.

59. *A fixed point theorem.* Suppose that a function f is continuous on the closed interval [0, 1] and that $0 \le f(x) \le 1$ for every x in [0, 1]. Show that there must exist a number c in [0, 1] such that $f(c) = c$ (c is called a **fixed point** of f).

60. *The sign-preserving property of continuous functions.* Let f be defined on an interval (a, b) and suppose that $f(c) \neq 0$ at some c where f is continuous. Show that there is an interval $(c - \delta,\ c + \delta)$ about c where f has the same sign as $f(c)$. Notice how remarkable this conclusion is. Although f is defined throughout (a, b), it is not required to be continuous at any point except c. That and the condition $f(c) \neq 0$ are enough to make f different from zero (positive or negative) throughout an entire interval.

61. Explain how Theorem 6 follows from Theorem 1 in Section 1.2.

62. Explain how Theorem 7 follows from Theorems 2 and 3 in Section 1.2.

Solving Equations Graphically

Use a graphing calculator or computer grapher to solve the equations in Exercises 63–70.

63. $x^3 - 3x - 1 = 0$

64. $2x^3 - 2x^2 - 2x + 1 = 0$

65. $x(x - 1)^2 = 1$ (one root)

66. $x^x = 2$

67. $\sqrt{x} + \sqrt{1 + x} = 4$

68. $x^3 - 15x + 1 = 0$ (three roots)

69. $\cos x = x$ (one root). Make sure you are using radian mode.

70. $2 \sin x = x$ (three roots). Make sure you are using radian mode.

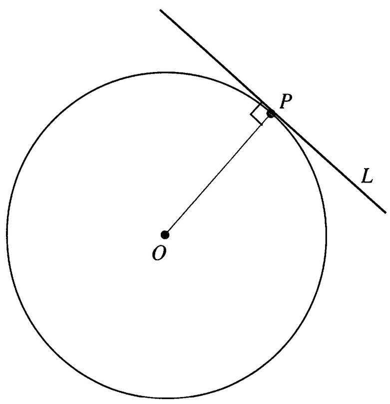
1.6 Tangent Lines

This section continues the discussion of secants and tangents begun in Section 1.1. We calculate limits of secant slopes to find tangents to curves.

What *Is* a Tangent to a Curve?

For circles, tangency is straightforward. A line L is tangent to a circle at a point P if L passes through P perpendicular to the radius at P (Fig. 1.48). Such a line just *touches* the circle. But what does it mean to say that a line L is tangent to some other curve C at a point P? Generalizing from the geometry of the circle, we might say that it means one of the following.

1. L passes through P perpendicular to the line from P to the center of C.
2. L passes through only one point of C, namely P.
3. L passes through P and lies on one side of C only.



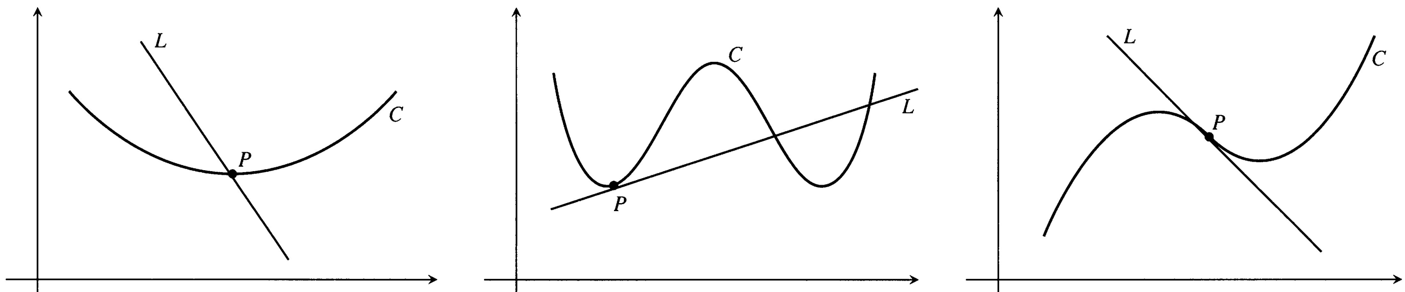
1.48 L is tangent to the circle at P if it passes through P perpendicular to radius OP.

While these statements are valid if C is a circle, none of them work consistently for more general curves. Most curves do not have centers, and a line we may want to call tangent may intersect C at other points or cross C at the point of tangency (Fig. 1.49 on the following page).

To define tangency for general curves, we need a dynamic approach that takes into account the behavior of the secants through P and nearby points Q as Q moves toward P along the curve (Fig. 1.50 on the following page). It goes like this:

1. We start with what we *can* calculate, namely the slope of the secant PQ.
2. Investigate the limit of the secant slope as Q approaches P along the curve.
3. If the limit exists, take it to be the slope of the curve at P and define the tangent to the curve at P to be the line through P with this slope.

This is what we were doing in the fruit fly example in Section 1.1.

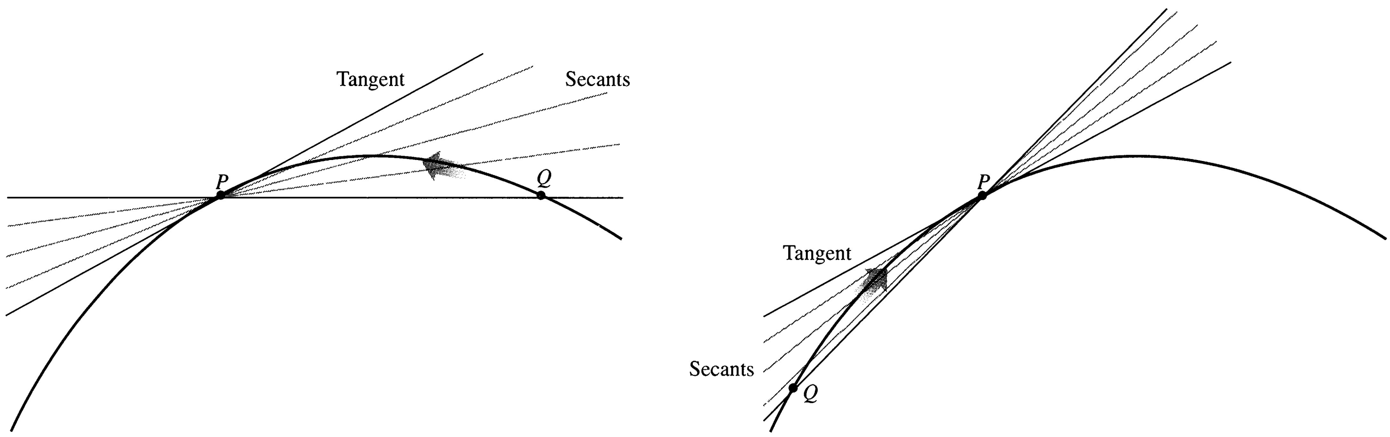


L meets C only at P but is not tangent to C.

L is tangent to C at P but meets C at several points.

L is tangent to C at P but lies on two sides of C, crossing C at P.

1.49 Exploding myths about tangent lines.



1.50 The dynamic approach to tangency. The tangent to the curve at P is the line through P whose slope is the limit of the secant slopes as $Q \to P$ from either side.

How do you find a tangent to a curve?

This was the dominant mathematical question of the early seventeenth century and it is hard to overestimate how badly the scientists of the day wanted to know the answer. In optics, the tangent determined the angle at which a ray of light entered a curved lens. In mechanics, the tangent determined the direction of a body's motion at every point along its path. In geometry, the tangents to two curves at a point of intersection determined the angle at which the curves intersected. Descartes went so far as to say that the problem of finding a tangent to a curve was "the most useful and most general problem not only that I know but even that I have any desire to know."

EXAMPLE 1 Find the slope of the parabola $y = x^2$ at the point $P(2, 4)$. Write an equation for the tangent to the parabola at this point.

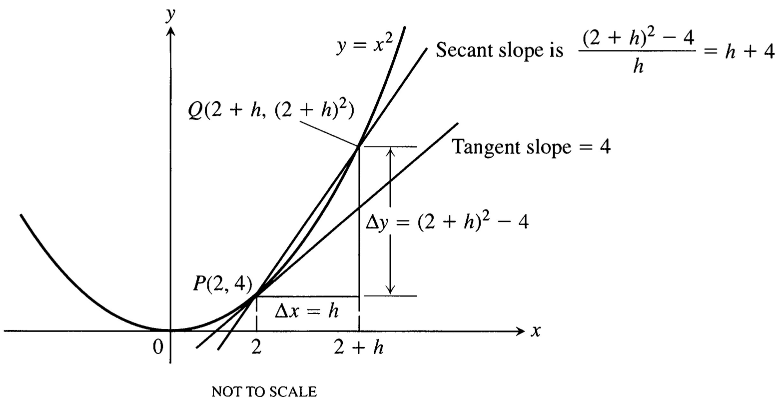
Solution We begin with a secant line through $P(2, 4)$ and $Q(2+h, (2+h)^2)$ nearby. We then write an expression for the slope of the secant PQ and investigate what happens to the slope as Q approaches P along the curve:

$$\begin{aligned}\text{Secant slope} &= \frac{\Delta y}{\Delta x} = \frac{(2+h)^2 - 2^2}{h} = \frac{h^2 + 4h + 4 - 4}{h} \\ &= \frac{h^2 + 4h}{h} = h + 4.\end{aligned}$$

If $h > 0$, Q lies above and to the right of P, as in Fig. 1.51. If $h < 0$, Q lies to the left of P (not shown). In either case, as Q approaches P along the curve, h approaches zero and the secant slope approaches 4:

$$\lim_{h \to 0} (h + 4) = 4.$$

We take 4 to be the parabola's slope at P.

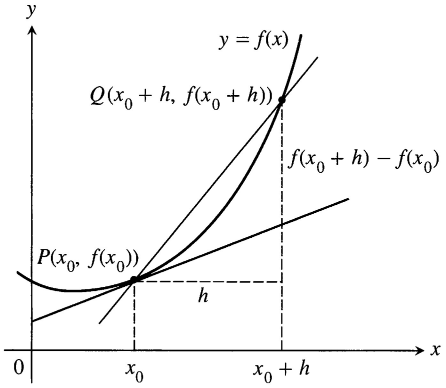


1.51 Diagram for finding the slope of the parabola $y = x^2$ at the point $P(2, 4)$ (Example 1).

The tangent to the parabola at P is the line through P with slope 4:

$$y = 4 + 4(x - 2) \quad \text{Point-slope equation}$$

$$y = 4x - 4. \qquad \square$$



1.52 The tangent slope is

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

Finding a Tangent to the Graph of a Function

To find a tangent to an arbitrary curve $y = f(x)$ at a point $P(x_0, f(x_0))$ we use the same dynamic procedure. We calculate the slope of the secant through P and a point $Q(x_0 + h, f(x_0 + h))$. We then investigate the limit of the slope as $h \to 0$ (Fig. 1.52). If the limit exists, we call it the slope of the curve at P and define the tangent at P to be the line through P having this slope.

Definitions

The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \quad \text{(provided the limit exists).}$$

The **tangent line** to the curve at P is the line through P with this slope.

Whenever we make a new definition it is a good idea to try it on familiar objects to be sure it gives the results we want in familiar cases. The next example shows that the new definition of slope agrees with the old definition when we apply it to nonvertical lines.

How to Find the Tangent to the Curve $y = f(x)$ at (x_0, y_0)

1. Calculate $f(x_0)$ and $f(x_0 + h)$.
2. Calculate the slope
$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$
3. If the limit exists, find the tangent line as $y = y_0 + m(x - x_0)$.

EXAMPLE 2 *Testing the definition*

Show that the line $y = mx + b$ is its own tangent at any point $(x_0, mx_0 + b)$.

Solution We let $f(x) = mx + b$ and organize the work into three steps.

Step 1: *Find $f(x_0)$ and $f(x_0 + h)$.*

$$f(x_0) = mx_0 + b$$

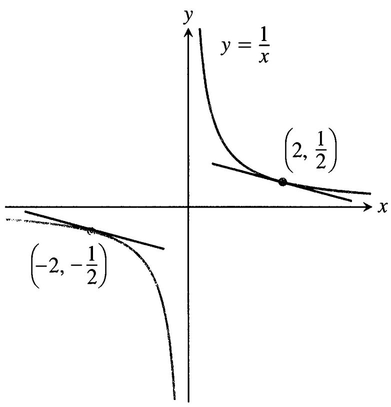
$$f(x_0 + h) = m(x_0 + h) + b = mx_0 + mh + b$$

Pierre de Fermat (1601–1665)

The dynamic approach to tangency, invented by Fermat in 1629, proved to be one of the seventeenth century's major contributions to calculus.

Fermat, a skilled linguist and one of his century's greatest mathematicians, tended to confine his writing to professional correspondence and to papers written for personal friends. He rarely wrote completed descriptions of his work, even for his personal use. His famous "last theorem" (that $a^n + b^n = c^n$ has no positive integer solutions for a, b, and c if n is an integer greater than 2) is known only from a note he jotted in the margin of a book. His name slipped into relative obscurity until the late 1800s, and it was only from a four-volume edition of his works published at the beginning of this century that the true importance of his many achievements became clear.

Besides the work in physics and number theory for which he is best known, Fermat found the areas under curves as limits of sums of rectangle areas (as we do today) and developed a method for finding the centroids of shapes bounded by curves in the plane. The standard formula for the first derivative of a polynomial function, the formulas for calculating arc length and for finding the area of a surface of revolution, and the second derivative test for extreme values of functions can all be found in his papers. We will see what these are as the text continues.



1.53 The two tangent lines to $y = 1/x$ having slope $-1/4$.

Step 2: *Find the slope* $\lim_{h\to 0} (f(x_0 + h) - f(x_0))/h$.

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h\to 0} \frac{(mx_0 + mh + b) - (mx_0 + b)}{h}$$
$$= \lim_{h\to 0} \frac{mh}{h} = m$$

Step 3: *Find the tangent line using the point–slope equation.* The tangent line at the point $(x, mx_0 + b)$ is

$$y = (mx_0 + b) + m(x - x_0)$$
$$y = mx_0 + b + mx - mx_0$$
$$y = mx + b.$$

❑

EXAMPLE 3

a) Find the slope of the curve $y = 1/x$ at $x = a$.
b) Where does the slope equal $-1/4$?
c) What happens to the tangent to the curve at the point $(a, 1/a)$ as a changes?

Solution

a) Here $f(x) = 1/x$. The slope at $(a, 1/a)$ is

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h}$$
$$= \lim_{h\to 0} \frac{1}{h}\frac{a - (a+h)}{a(a+h)}$$
$$= \lim_{h\to 0} \frac{-h}{ha(a+h)}$$
$$= \lim_{h\to 0} \frac{-1}{a(a+h)} = -\frac{1}{a^2}.$$

Notice how we had to keep writing "$\lim_{h\to 0}$" at the beginning of each line until the stage where we could evaluate the limit by substituting $h = 0$.

b) The slope of $y = 1/x$ at the point where $x = a$ is $-1/a^2$. It will be $-1/4$ provided

$$-\frac{1}{a^2} = -\frac{1}{4}.$$

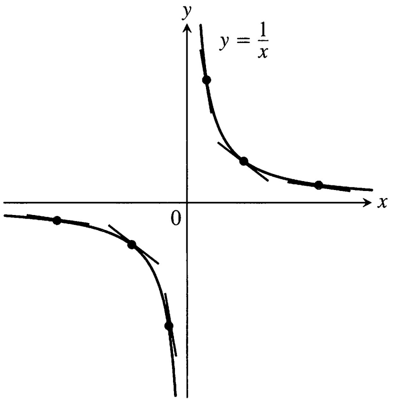
This equation is equivalent to $a^2 = 4$, so $a = 2$ or $a = -2$. The curve has slope $-1/4$ at the two points $(2, 1/2)$ and $(-2, -1/2)$ (Fig. 1.53).

c) Notice that the slope $-1/a^2$ is always negative. As $a \to 0^+$, the slope approaches $-\infty$ and the tangent becomes increasingly steep (Fig. 1.54). We see this again as $a \to 0^-$. As a moves away from the origin, the slope approaches 0^- and the tangent levels off. ❑

Rates of Change

The expression

$$\frac{f(x_0 + h) - f(x_0)}{h}$$



1.54 The tangent slopes, steep near the origin, become more gradual as the point of tangency moves away.

All of these refer to the same thing.

1. The slope of $y = f(x)$ at $x = x_0$
2. The slope of the tangent to $y = f(x)$ at $x = x_0$
3. The rate of change of $f(x)$ with respect to x at $x = x_0$
4. The derivative of f at $x = x_0$
5. $\lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$

is called the **difference quotient of f at x_0**. If the difference quotient has a limit as h approaches zero, that limit is called the **derivative of f at x_0**. If we interpret the difference quotient as a secant slope, the derivative gives the slope of the curve and tangent at the point where $x = x_0$. If we interpret the difference quotient as an average rate of change, as we did in Section 1.1, the derivative gives the function's rate of change with respect to x at the point $x = x_0$. The derivative is one of the two most important mathematical objects considered in calculus. We will begin a thorough study of it in Chapter 2.

EXAMPLE 4 *Instantaneous speed (Continuation of Section 1.1, Examples 1 and 2)*

In Examples 1 and 2 in Section 1.1, we studied the speed of a rock falling freely from rest near the surface of the earth. We knew that the rock fell $y = 16t^2$ feet during the first t seconds, and we used a sequence of average rates over increasingly short intervals to estimate the rock's speed at the instant $t = 1$. Exactly what *was* the rock's speed at this time?

Solution We let $f(t) = 16t^2$. The average speed of the rock over the interval between $t = 1$ and $t = 1 + h$ seconds was

$$\frac{f(1+h) - f(1)}{h} = \frac{16(1+h)^2 - 16(1)^2}{h} = \frac{16(h^2+2h)}{h} = 16(h+2).$$

The rock's speed at the instant $t = 1$ was

$$\lim_{h\to 0} 16(h+2) = 16(0+2) = 32 \text{ ft/sec}.$$

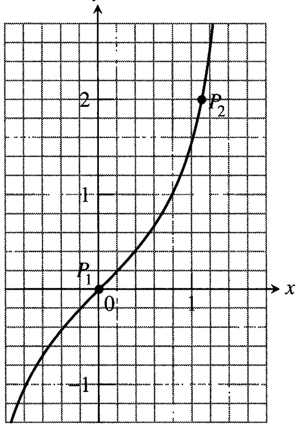
Our original estimate of 32 ft/sec was right. ❑

Exercises 1.6

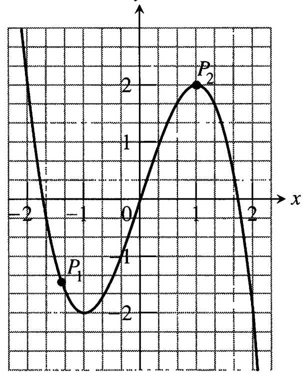
Slopes and Tangent Lines

In Exercises 1–4, use the grid and a straight edge to make a rough estimate of the slope of the curve (in y-units per x-unit) at the points P_1 and P_2. Graphs can shift during a press run, so your estimates may be somewhat different from those in the back of the book.

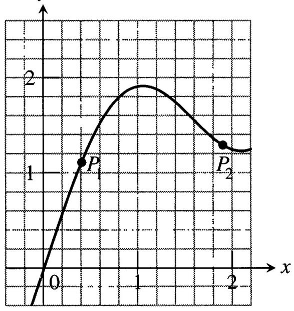
1.



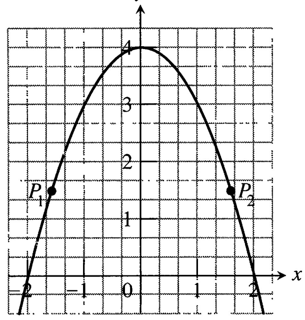
2.



3.



4.



In Exercises 5–10, find an equation for the tangent to the curve at the given point. Then sketch the curve and tangent together.

5. $y = 4 - x^2$, $(-1, 3)$

6. $y = (x-1)^2 + 1$, $(1, 1)$

7. $y = 2\sqrt{x}$, $(1, 2)$

8. $y = \dfrac{1}{x^2}$, $(-1, 1)$

9. $y = x^3$, $(-2, -8)$

10. $y = \dfrac{1}{x^3}$, $\left(-2, -\dfrac{1}{8}\right)$

In Exercises 11–18, find the slope of the function's graph at the given point. Then find an equation for the line tangent to the graph there.

11. $f(x) = x^2 + 1$, $(2, 5)$

12. $f(x) = x - 2x^2$, $(1, -1)$

13. $g(x) = \dfrac{x}{x-2}$, $(3, 3)$

14. $g(x) = \dfrac{8}{x^2}$, $(2, 2)$

15. $h(t) = t^3$, $(2, 8)$

16. $h(t) = t^3 + 3t$, $(1, 4)$

17. $f(x) = \sqrt{x}$, $(4, 2)$

18. $f(x) = \sqrt{x+1}$, $(8, 3)$

In Exercises 19–22, find the slope of the curve at the point indicated.

19. $y = 5x^2$, $x = -1$

20. $y = 1 - x^2$, $x = 2$

21. $y = \dfrac{1}{x-1}$, $x = 3$

22. $y = \dfrac{x-1}{x+1}$, $x = 0$

Tangent Lines with Specified Slopes

At what points do the graphs of the functions in Exercises 23 and 24 have horizontal tangents?

23. $f(x) = x^2 + 4x - 1$

24. $g(x) = x^3 - 3x$

25. Find equations of all lines having slope -1 that are tangent to the curve $y = 1/(x-1)$.

26. Find an equation of the straight line having slope $1/4$ that is tangent to the curve $y = \sqrt{x}$.

Rates of Change

27. An object is dropped from the top of a 100-m-high tower. Its height aboveground after t seconds is $100 - 4.9t^2$ m. How fast is it falling 2 sec after it is dropped?

28. At t seconds after lift-off, the height of a rocket is $3t^2$ ft. How fast is the rocket climbing after 10 sec?

29. What is the rate of change of the area of a circle ($A = \pi r^2$) with respect to its radius when the radius is $r = 3$?

30. What is the rate of change of the volume of a ball ($V = (4/3)\pi r^3$) with respect to the radius when the radius is $r = 2$?

Testing for Tangents

31. Does the graph of

$$f(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

have a tangent at the origin? Give reasons for your answer.

32. Does the graph of

$$g(x) = \begin{cases} x \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

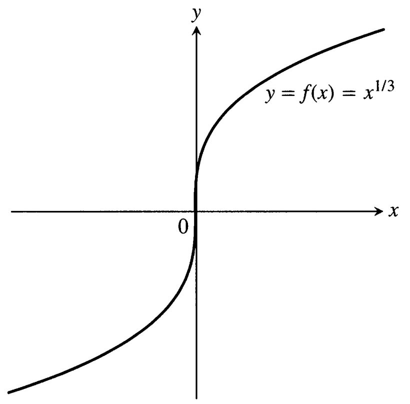
have a tangent at the origin? Give reasons for your answer.

Vertical Tangents

We say that the curve $y = f(x)$ has a **vertical tangent** at the point where $x = x_0$ if $\lim_{h\to 0} (f(x_0+h) - f(x_0))/h = \infty$ or $-\infty$.

Vertical tangent at $x = 0$:

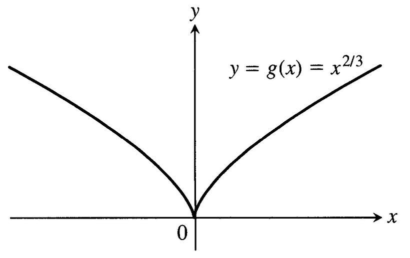
$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{h^{1/3} - 0}{h} = \lim_{h\to 0} \frac{1}{h^{2/3}} = \infty$$



No vertical tangent at $x = 0$:

$$\lim_{h\to 0} \frac{g(0+h) - g(0)}{h} = \lim_{h\to 0} \frac{h^{2/3} - 0}{h} = \lim_{h\to 0} \frac{1}{h^{1/3}}$$

does not exist, because the limit is ∞ from the right and $-\infty$ from the left.



33. Does the graph of

$$f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

have a vertical tangent at the origin? Give reasons for your answer.

34. Does the graph of

$$U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

have a vertical tangent at the point (0, 1)? Give reasons for your answer.

Grapher Explorations—Vertical Tangents

a) Graph the curves in Exercises 35–44. Where do the graphs appear to have vertical tangents?
b) Confirm your findings in (a) with limit calculations.

35. $y = x^{2/5}$

36. $y = x^{4/5}$

37. $y = x^{1/5}$

38. $y = x^{3/5}$

39. $y = 4x^{2/5} - 2x$

40. $y = x^{5/3} - 5x^{2/3}$

41. $y = x^{2/3} - (x - 1)^{1/3}$

42. $y = x^{1/3} + (x - 1)^{1/3}$

43. $y = \begin{cases} -\sqrt{|x|}, & x \leq 0 \\ \sqrt{x}, & x > 0 \end{cases}$

44. $y = \sqrt{|4 - x|}$

CAS Explorations and Projects

Use a CAS to perform the following steps for the functions in Exercises 45–48.

a) Plot $y = f(x)$ over the interval $x_0 - \frac{1}{2} \leq x \leq x_0 + 3$.
b) Define the difference quotient q at x_0 as a function of the general step size h.
c) Find the limit of q as $h \to 0$.
d) Define the secant lines $y = f(x_0) + q^*(x - x_0)$ for $h = 3, 2,$ and 1. Graph them together with f and the tangent line over the interval in part (a).

45. $f(x) = x^3 + 2x, \quad x_0 = 0$

46. $f(x) = x + \frac{5}{x}, \quad x_0 = 1$

47. $f(x) = x + \sin\ (2x), \quad x_0 = \pi/2$

48. $f(x) = \cos\ x + 4\sin\ (2x), \quad x_0 = \pi$

Chapter 1 Questions to Guide Your Review

1. What is the average rate of change of the function $g(t)$ over the interval from $t = a$ to $t = b$? How is it related to a secant line?

2. What limit must be calculated to find the rate of change of a function $g(t)$ at $t = t_0$?

3. Does the existence and value of the limit of a function $f(x)$ as x approaches c ever depend on what happens at $x = c$? Explain, and give examples.

4. What theorems are available for calculating limits? Give examples of how the theorems are used.

5. How are one-sided limits related to limits? How can this relationship sometimes be used to calculate a limit or prove it does not exist? Give examples.

6. How is the problem of controlling the input x of a function f so that the output $y = f(x)$ will be within a certain specified tolerance ϵ of a target value $y_0 = f(x_0)$ related to the problem of proving that f has limit y_0 as $x \to x_0$?

7. What exactly does $\lim_{x \to x_0} f(x) = L$ mean? Give an example in which you find a $\delta > 0$ for a given $f, L, x_0,$ and $\epsilon > 0$ in the formal definition of limit.

8. Give formal definitions of the following statements.

a) $\lim_{x \to 2^-} f(x) = 5$
b) $\lim_{x \to 2^+} f(x) = 5$
c) $\lim_{x \to 2} f(x) = \infty$
d) $\lim_{x \to 2} f(x) = -\infty$

9. What conditions must be satisfied by a function if it is to be continuous at an interior point of its domain? at an endpoint?

10. How can looking at the graph of a function help you tell where the function is continuous?

11. What does it mean for a function to be right-continuous at a point? left-continuous? How are continuity and one-sided continuity related?

12. What can be said about the continuity of polynomials? of rational functions? of trigonometric functions? of rational powers and algebraic combinations of functions? of composites of functions? of absolute values of functions?

13. Under what circumstances can you extend a function $f(x)$ to be continuous at a point $x = c$? Give an example.

14. What does it mean for a function to be continuous on an interval?

15. What does it mean for a function to be continuous? Give examples to illustrate the fact that a function that is not continuous on its entire domain may still be continuous on selected intervals within the domain.

CHAPTER 2

Derivatives

OVERVIEW In Chapter 1 we defined the slope of a curve at a point as the limit of secant slopes. This limit, called a derivative, measures the rate at which a function changes and is one of the most important ideas in calculus. Derivatives are used widely in science, economics, medicine, and computer science to calculate velocity and acceleration, to explain the behavior of machinery, to estimate the drop in water levels as water is pumped out of a tank, and to predict the consequences of making errors in measurements. Finding derivatives by evaluating limits can be lengthy and difficult. In this chapter we develop techniques to make calculating derivatives easier.

2.1 The Derivative of a Function

At the end of Chapter 1, we defined the slope of a curve $y = f(x)$ at the point where $x = x_0$ to be

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

We called this limit, when it existed, the derivative of f at x_0. In this section, we investigate the derivative as a *function* derived from f by considering the limit at each point of f's domain.

Definition

The **derivative** of the function f with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h},$$

provided the limit exists.

The domain of f', the set of points in the domain of f for which the limit exists, may be smaller than the domain of f. If $f'(x)$ exists, we say that f **has a derivative (is differentiable)** at x.

Notation

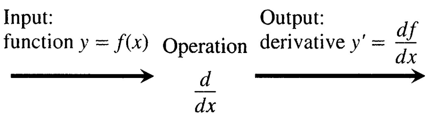
There are many ways to denote the derivative of a function $y = f(x)$. Besides $f'(x)$, the most common notations are these:

y'	"y prime"	Nice and brief but does not name the independent variable
$\dfrac{dy}{dx}$	"$dy\ \ dx$"	Names the variables and uses d for derivative
$\dfrac{df}{dx}$	"$df\ \ dx$"	Emphasizes the function's name
$\dfrac{d}{dx}f(x)$	"ddx of $f(x)$"	Emphasizes the idea that differentiation is an operation performed on f (Fig. 2.1)
$D_x f$	"dx of f"	A common operator notation
$\dot{y}$	"y dot"	One of Newton's notations, now common for time derivatives

We also read dy/dx as "the derivative of y with respect to x," and df/dx and $(d/dx)f(x)$ as "the derivative of f with respect to x."

Why all these notations?

The "prime" notations y' and f' come from notations that Newton used for derivatives. The d/dx notations are similar to those used by Leibniz. Each has its own strengths and weaknesses.



2.1 Flow diagram for the operation of taking a derivative with respect to x.

Calculating Derivatives from the Definition

The process of calculating a derivative is called **differentiation.** Examples 2 and 3 of Section 1.6 illustrate the process for the functions $y = mx + b$ and $y = 1/x$. Example 2 shows that

$$\frac{d}{dx}(mx + b) = m.$$

In Example 3, we see that

$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}.$$

Here are two more examples.

Steps for Calculating $f'(x)$ from the Definition of Derivative

1. Write expressions for $f(x)$ and $f(x+h)$.
2. Expand and simplify the difference quotient
$$\frac{f(x+h) - f(x)}{h}.$$
3. Using the simplified quotient, find $f'(x)$ by evaluating the limit
$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

EXAMPLE 1

a) Differentiate $f(x) = \dfrac{x}{x-1}$.

b) Where does the curve $y = f(x)$ have slope -1?

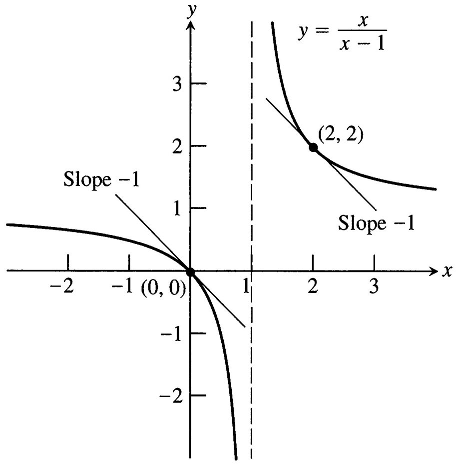
Solution

a) We take the three steps listed in the margin.

Step 1: Here we have $f(x) = \dfrac{x}{x-1}$

and

$$f(x+h) = \frac{(x+h)}{(x+h)-1}, \text{ so}$$



2.2 $y' = -1$ at $x = 0$ and $x = 2$.

Step 2: $$\frac{f(x+h)-f(x)}{h} = \frac{\dfrac{x+h}{x+h-1} - \dfrac{x}{x-1}}{h}$$

$$= \frac{1}{h} \cdot \frac{(x+h)(x-1) - x(x+h-1)}{(x+h-1)(x-1)} \qquad \frac{a}{b} - \frac{c}{d} = \frac{ad-cb}{bd}$$

$$= \frac{1}{h} \cdot \frac{-h}{(x+h-1)(x-1)}, \text{ and}$$

Step 3: $$f'(x) = \lim_{h\to 0} \frac{-1}{(x+h-1)(x-1)} = \frac{-1}{(x-1)^2}.$$

b) The slope of $y = f(x)$ will be -1 provided

$$-\frac{1}{(x-1)^2} = -1.$$

This equation is equivalent to $(x-1)^2 = 1$, so $x = 2$ or $x = 0$ (Fig. 2.2). ❑

EXAMPLE 2

a) Find the derivative of $y = \sqrt{x}$ for $x > 0$.
b) Find the tangent line to the curve $y = \sqrt{x}$ at $x = 4$.

Solution

You will often need to know the derivative of $\sqrt{x}$ for $x > 0$:

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}.$$

Try to remember it.

---

a) ***Step 1:*** $f(x) = \sqrt{x}$ and $f(x+h) = \sqrt{x+h}$

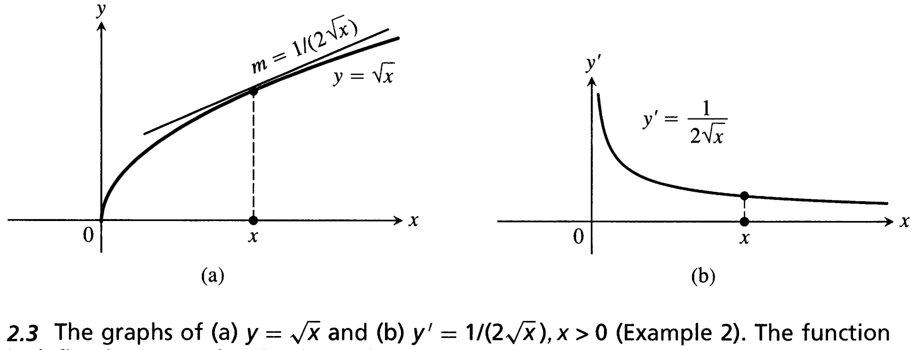
Step 2: $$\frac{f(x+h)-f(x)}{h} = \frac{\sqrt{x+h}-\sqrt{x}}{h} \qquad \text{Multiply by } \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$= \frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})}$$

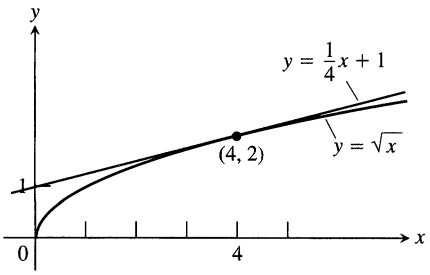
$$= \frac{1}{\sqrt{x+h}+\sqrt{x}}$$

Step 3: $$f'(x) = \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

See Fig. 2.3.



2.3 The graphs of (a) $y = \sqrt{x}$ and (b) $y' = 1/(2\sqrt{x}), x > 0$ (Example 2). The function is defined at $x = 0$, but its derivative is not.



2.4 The curve $y = \sqrt{x}$ and its tangent at (4, 2). The tangent's slope is found by evaluating dy/dx at $x = 4$ (Example 2).

The symbol for evaluation

In addition to

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

the value of the derivative of $y = f(x)$ with respect to x at $x = a$ can be denoted in the following ways:

$$y'\,|_{x=a} = \left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{d}{dx} f(x)\right|_{x=a}.$$

Here the symbol $|_{x=a}$, called an **evaluation symbol,** tells us to evaluate the expression to its left at $x = a$.

b) The slope of the curve at $x = 4$ is

$$\left.\frac{dy}{dx}\right|_{x=4} = \left.\frac{1}{2\sqrt{x}}\right|_{x=4} = \frac{1}{4}.$$

The tangent is the line through the point (4, 2) with slope 1/4 (Fig. 2.4).

$$y = 2 + \frac{1}{4}(x - 4)$$

$$y = \frac{1}{4}x + 1$$

❑

Graphing f' from Estimated Values

When we measure the values of a function $y = f(x)$ in the laboratory or in the field (pressure vs. temperature, say, or population vs. time) we usually connect the data points with lines or curves to picture the graph of f. We can often make a reasonable plot of f' by estimating slopes on this graph. The following examples show how this is done and what can be learned from the process.

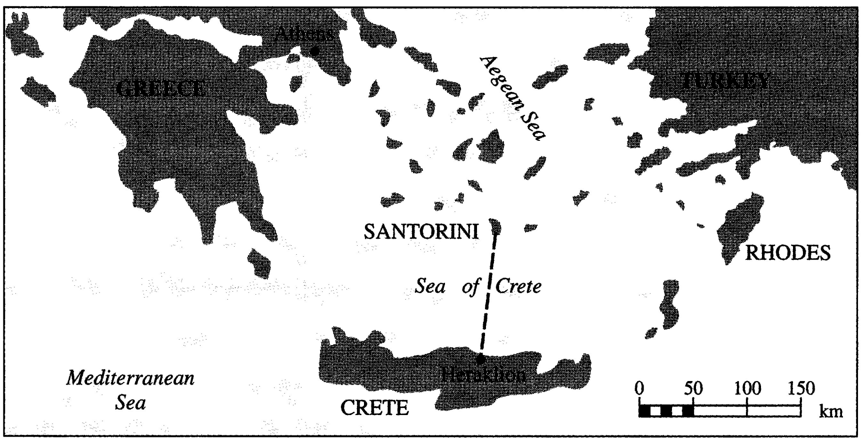
EXAMPLE 3 *Medicine*

On April 23, 1988, the human-powered airplane *Daedalus* flew a record-breaking 119 km from Crete to the island of Santorini in the Aegean Sea, southeast of mainland Greece. During the 6-h endurance tests before the flight, researchers monitored the prospective pilots' blood-sugar concentrations. The concentration graph for one of the athlete-pilots is shown in Fig. 2.5(a), where the concentration in milligrams/deciliter is plotted against time in hours.

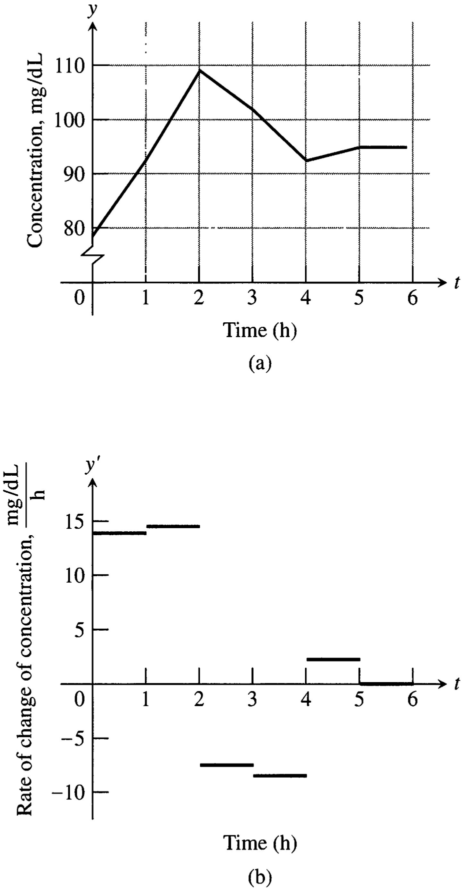
The graph is made of line segments connecting data points. The constant slope of each segment gives an estimate of the derivative of the concentration between measurements. We calculated the slope of each segment from the coordinate grid and plotted the derivative as a step function in Fig. 2.5(b). To make the plot for the first hour, for instance, we observed that the concentration increased from about 79 mg/dL to 93 mg/dL. The net increase was $\Delta y = 93 - 79 = 14$ mg/dL. Dividing this by $\Delta t = 1$ h gave the rate of change as

$$\frac{\Delta y}{\Delta t} = \frac{14}{1} = 14 \text{ mg/dL per h.}$$

❑



Daedalus's flight path on April 23, 1988.



2.5 (a) The sugar concentration in the blood of a *Daedalus* pilot during a 6-h preflight endurance test. (b) The derivative of the pilot's blood-sugar concentration shows how rapidly the concentration rose and fell during various portions of the test. (Source: *The Daedalus Project: Physiological Problems and Solutions* by Ethan R. Nadel and Steven R. Bussolari, *American Scientist,* Vol. 76, No. 4, July–August 1988, p. 358.)

Notice that we can make no estimate of the concentration's rate of change at times $t = 1, 2, \ldots, 5$, where the graph we have drawn for the concentration has a corner and no slope. The derivative step function is not defined at these times.

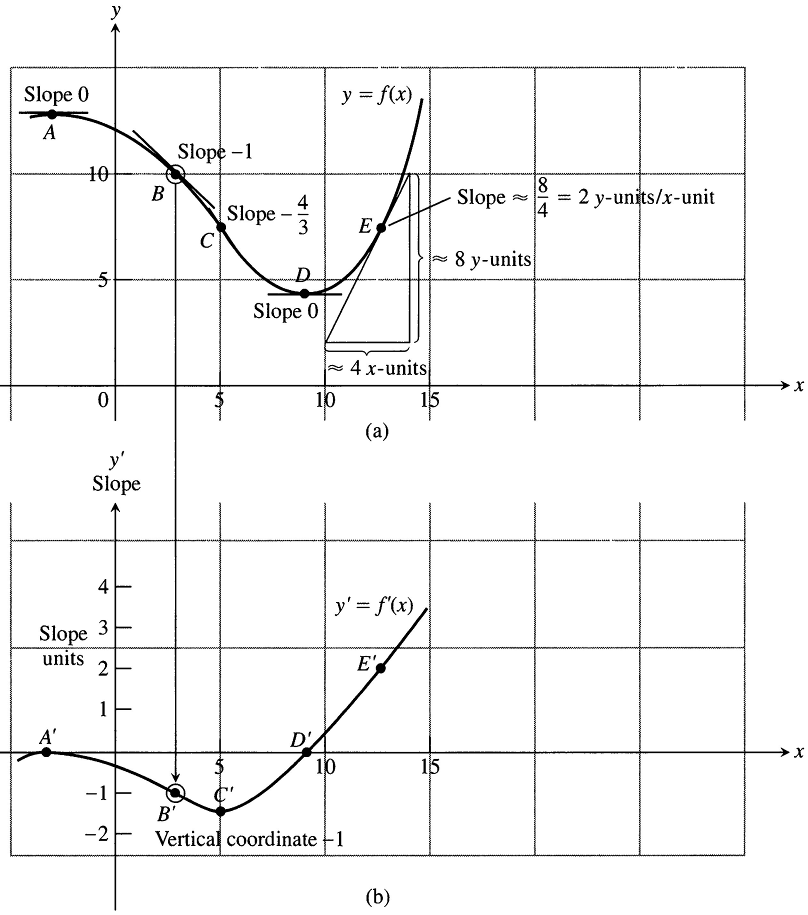
When we have so many data that the graph we get by connecting the data points resembles a smooth curve, we may wish to plot the derivative as a smooth curve. The next example shows how this is done.

EXAMPLE 4 Graph the derivative of the function $y = f(x)$ in Fig. 2.6(a).

Solution We draw a pair of axes, marking the horizontal axis in x-units and the vertical axis in y'-units (Fig. 2.6b). Next we sketch tangents to the graph of f at frequent intervals and use their slopes to estimate the values of $y' = f'(x)$ at these points. We plot the corresponding (x, y') pairs and connect them with a smooth curve.

From the graph of $y' = f'(x)$ we see at a glance

1. where f's rate of change is positive, negative, or zero;
2. the rough size of the growth rate at any x and its size in relation to the size of $f(x)$;
3. where the rate of change itself is increasing or decreasing. ❑



2.6 We made the graph of $y' = f'(x)$ in (b) by plotting slopes from the graph of $y = f(x)$ in (a). The vertical coordinate of B' is the slope at B, and so on. The graph of $y' = f'(x)$ is a visual record of how the slope of f changes with x.

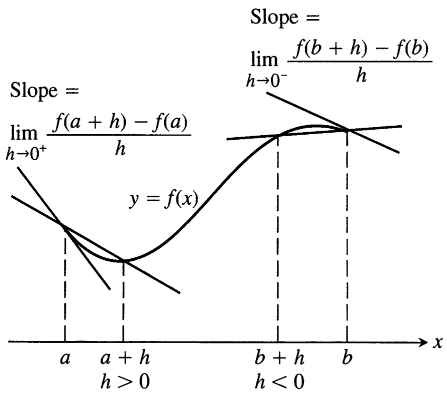
Differentiable on an Interval; One-sided Derivatives

A function $y = f(x)$ is **differentiable** on an open interval (finite or infinite) if it has a derivative at each point of the interval. It is differentiable on a closed interval $[a, b]$ if it is differentiable on the interior (a, b) and if the limits

$$\lim_{h\to 0^+} \frac{f(a+h) - f(a)}{h} \qquad \textbf{Right-hand derivative at } a$$

$$\lim_{h\to 0^-} \frac{f(b+h) - f(b)}{h} \qquad \textbf{Left-hand derivative at } b$$

exist at the endpoints (Fig. 2.7).



2.7 Derivatives at endpoints are one-sided limits.

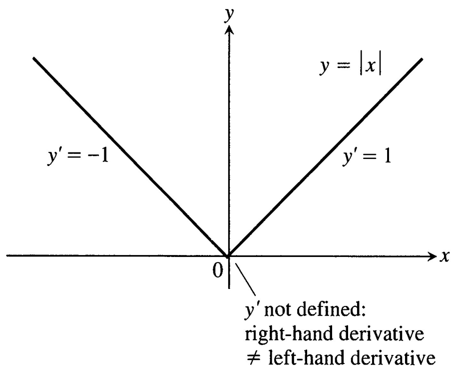
Right-hand and left-hand derivatives may be defined at any point of a function's domain. The usual relation between one-sided and two-sided limits holds for these derivatives. Because of Theorem 5, Section 1.4, a function has a derivative at a point if and only if it has left-hand and right-hand derivatives there, and these one-sided derivatives are equal.

EXAMPLE 5 The function $y = |x|$ is differentiable on $(-\infty, 0)$ and $(0, \infty)$ but has no derivative at $x = 0$. To the right of the origin,

$$\frac{d}{dx}(|x|) = \frac{d}{dx}(x) = \frac{d}{dx}(1 \cdot x) = 1. \qquad \frac{d}{dx}(mx + b) = m$$

To the left,

$$\frac{d}{dx}(|x|) = \frac{d}{dx}(-x) = \frac{d}{dx}(-1 \cdot x) = -1$$



2.8 Not differentiable at the origin.

(Fig. 2.8). There can be no derivative at the origin because the one-sided derivatives differ there:

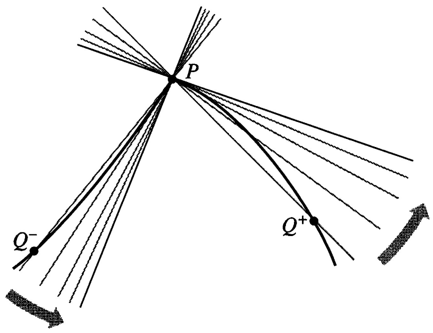
$$\begin{aligned}
\text{Right-hand derivative of } |x| \text{ at zero} &= \lim_{h\to 0^+} \frac{|0+h| - |0|}{h} = \lim_{h\to 0^+} \frac{|h|}{h} \\
&= \lim_{h\to 0^+} \frac{h}{h} \qquad |h| = h \text{ when } h > 0 \\
&= \lim_{h\to 0^+} 1 = 1 \\
\text{Left-hand derivative of } |x| \text{ at zero} &= \lim_{h\to 0^-} \frac{|0+h| - |0|}{h} = \lim_{h\to 0^-} \frac{|h|}{h} \\
&= \lim_{h\to 0^-} \frac{-h}{h} \qquad |h| = -h \text{ when } h < 0 \\
&= \lim_{h\to 0^-} -1 = -1.
\end{aligned}$$

❑

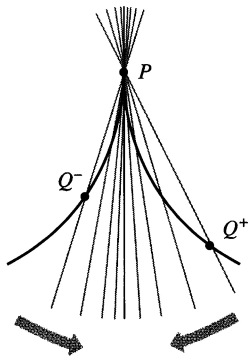
When Does a Function *Not* Have a Derivative at a Point?

A function has a derivative at a point x_0 if the slopes of the secant lines through $P(x_0, f(x_0))$ and a nearby point Q on the graph approach a limit as Q approaches P. Whenever the secants fail to take up a limiting position or become vertical as Q approaches P, the derivative does not exist. A function whose graph is otherwise smooth will fail to have a derivative at a point where the graph has

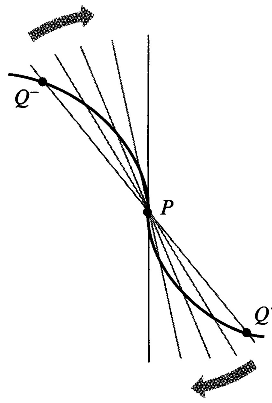
1. a *corner,* where the one-sided derivatives differ



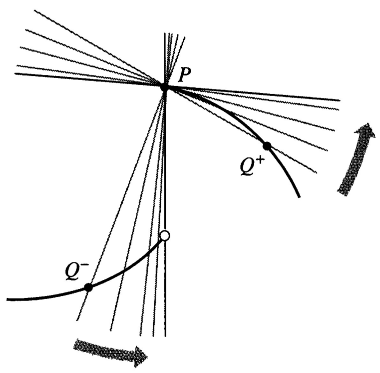
2. a *cusp,* where the slope of PQ approaches ∞ from one side and $-\infty$ from the other

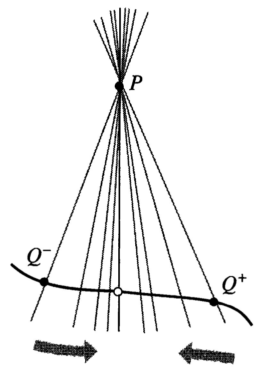


3. a *vertical tangent,* where the slope of PQ approaches ∞ from both sides or approaches $-\infty$ from both sides (here, $-\infty$)



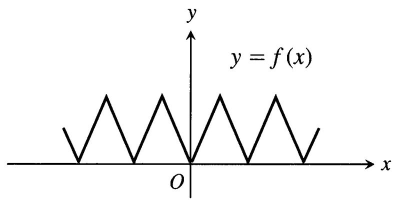
4. a *discontinuity.*





How rough can the graph of a continuous function be?

The absolute value function fails to be differentiable at a single point. Using a similar idea, we can use a sawtooth graph to define a continuous function that fails to have a derivative at infinitely many points.



But can a continuous function fail to have a derivative at *every* point?

The answer, surprisingly enough, is yes, as Karl Weierstrass (1815–1897) found in 1872. One of his formulas (there are many like it) was

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n \cos(9^n \pi x),$$

a formula that expresses f as an infinite sum of cosines with increasingly higher frequencies. By adding wiggles to wiggles infinitely many times, so to speak, the formula produces a graph that is too bumpy in the limit to have a tangent anywhere.

Continuous curves that fail to have a tangent anywhere play a useful role in chaos theory, in part because there is no way to assign a finite length to such a curve. We will see what length has to do with derivatives when we get to Section 5.5.

Differentiable Functions Are Continuous

A function is continuous at every point where it has a derivative.

Theorem 1

If f has a derivative at $x = c$, then f is continuous at $x = c$.

Proof Given that $f'(c)$ exists, we must show that $\lim_{x\to c} f(x) = f(c)$, or, equivalently, that $\lim_{h\to 0} f(c+h) = f(c)$. If $h \neq 0$, then

$$\begin{aligned} f(c+h) &= f(c) + (f(c+h) - f(c)) \\ &= f(c) + \frac{f(c+h) - f(c)}{h} \cdot h. \end{aligned}$$

Now take limits as $h \to 0$. By Theorem 1 of Section 1.2,

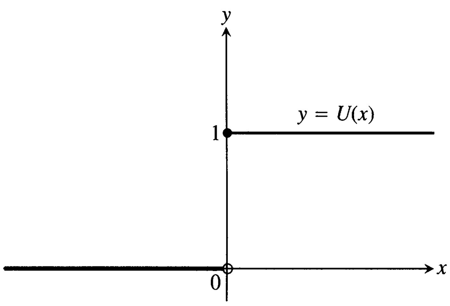
$$\begin{aligned} \lim_{h\to 0} f(c+h) &= \lim_{h\to 0} f(c) + \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} \cdot \lim_{h\to 0} h \\ &= f(c) + f'(c) \cdot 0 \\ &= f(c) + 0 \\ &= f(c). \end{aligned}$$ ❑

Similar arguments with one-sided limits show that if f has a derivative from one side (right or left) at $x = c$, then f is continuous from that side at $x = c$.

Caution The converse of Theorem 1 is false. A function need not have a derivative at a point where it is continuous, as we saw in Example 5.

The Intermediate Value Property of Derivatives

Not every function can be some function's derivative, as we see from the following theorem.



2.9 The unit step function does not have the intermediate value property and cannot be the derivative of a function on the real line.

Theorem 2

If a and b are any two points in an interval on which f is differentiable, then f' takes on every value between $f'(a)$ and $f'(b)$.

Theorem 2 (which we will not prove) says that a function cannot *be* a derivative on an interval unless it has the intermediate value property there (Fig. 2.9). The question of when a function is a derivative is one of the central questions in all calculus, and Newton's and Leibniz's answer to this question revolutionized the world of mathematics. We will see what their answer was when we reach Chapter 4.

Exercises 2.1

Finding Derivative Functions and Values

Using the definition, calculate the derivatives of the functions in Exercises 1–6. Then find the values of the derivatives as specified.

1. $f(x) = 4 - x^2;\quad f'(-3), f'(0), f'(1)$

2. $F(x) = (x-1)^2 + 1;\quad F'(-1), F'(0), F'(2)$

3. $g(t) = \dfrac{1}{t^2};\quad g'(-1), g'(2), g'(\sqrt{3})$

4. $k(z) = \dfrac{1-z}{2z};\quad k'(-1), k'(1), k'(\sqrt{2})$

5. $p(\theta) = \sqrt{3\theta};\quad p'(1), p'(3), p'(2/3)$

6. $r(s) = \sqrt{2s+1};\quad r'(0), r'(1), r'(1/2)$

In Exercises 7–12, find the indicated derivatives.

7. $\dfrac{dy}{dx}$ if $y = 2x^3$

8. $\dfrac{dr}{ds}$ if $r = \dfrac{s^3}{2} + 1$

9. $\dfrac{ds}{dt}$ if $s = \dfrac{t}{2t+1}$

10. $\dfrac{dv}{dt}$ if $v = t - \dfrac{1}{t}$

11. $\dfrac{dp}{dq}$ if $p = \dfrac{1}{\sqrt{q+1}}$

12. $\dfrac{dz}{dw}$ if $z = \dfrac{1}{\sqrt{3w-2}}$

Slopes and Tangent Lines

In Exercises 13–16, differentiate the functions and find the slope of the tangent line at the given value of the independent variable.

13. $f(x) = x + \dfrac{9}{x},\quad x = -3$

14. $k(x) = \dfrac{1}{2+x},\quad x = 2$

15. $s = t^3 - t^2,\quad t = -1$

16. $y = (x+1)^3,\quad x = -2$

In Exercises 17–18, differentiate the functions. Then find an equation of the tangent line at the indicated point on the graph of the function.

17. $y = f(x) = \dfrac{8}{\sqrt{x-2}},\quad (x, y) = (6, 4)$

18. $w = g(z) = 1 + \sqrt{4-z},\quad (z, w) = (3, 2)$

In Exercises 19–22, find the values of the derivatives.

19. $\left.\dfrac{ds}{dt}\right|_{t=-1}$ if $s = 1 - 3t^2$

20. $\left.\dfrac{dy}{dx}\right|_{x=\sqrt{3}}$ if $y = 1 - \dfrac{1}{x}$

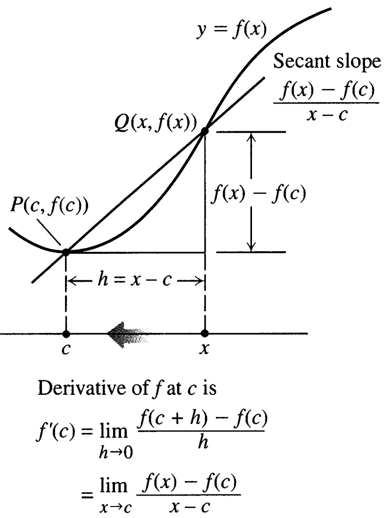
21. $\left.\dfrac{dr}{d\theta}\right|_{\theta=0}$ if $r = \dfrac{2}{\sqrt{4-\theta}}$

22. $\left.\dfrac{dw}{dz}\right|_{z=4}$ if $w = z + \sqrt{z}$

An Alternative Formula for Calculating Derivatives

The formula for the secant slope whose limit leads to the derivative depends on how the points involved are labeled. In the notation of Fig. 2.10, the secant slope is $(f(x) - f(c))/(x - c)$ and the slope of the curve at P is

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$



2.10 The way we write the difference quotient for the derivative of a function f depends on how we label the points involved.

The use of this formula simplifies some derivative calculations. Use it in Exercises 23–26 to find the derivative of the function at the given value of c.

23. $f(x) = \dfrac{1}{x+2},\quad c = -1$

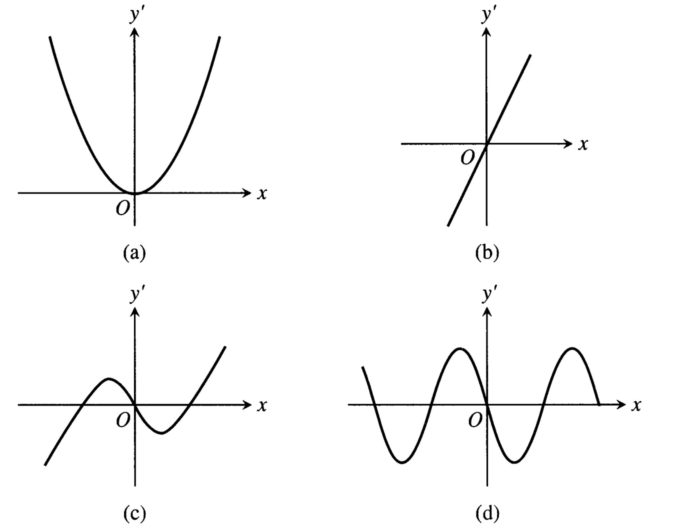
24. $f(x) = \dfrac{1}{(x-1)^2},\quad c = 2$

25. $g(t) = \dfrac{t}{t-1},\quad c = 3$

26. $k(s) = 1 + \sqrt{s},\quad c = 9$

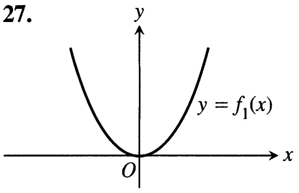
Graphs

Match the functions graphed in Exercises 27–30 with the derivatives graphed in Fig. 2.11.



2.11 The derivative graphs for Exercises 27–30.

27.



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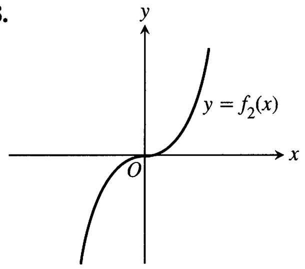

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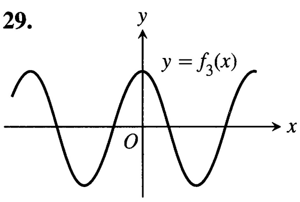
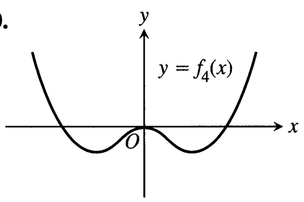
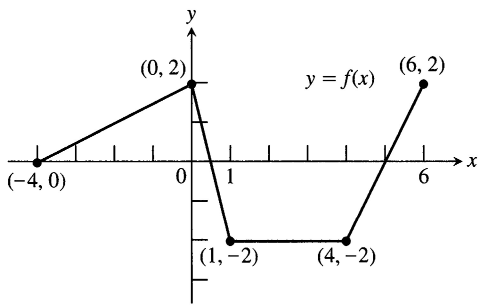


30.



31. a) The graph in Fig. 2.12 is made of line segments joined end to end. At which points of the interval $[-4, 6]$ is f' not defined? Give reasons for your answer.



2.12 The graph for Exercise 31.

b) Graph the derivative of f. Call the vertical axis the y'-axis. The graph should show a step function.

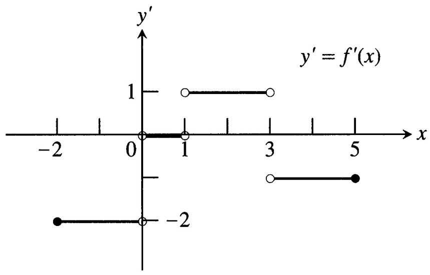
32. *Recovering a function from its derivative*

a) Use the following information to graph the function f over the closed interval $[-2, 5]$.

i) The graph of f is made of closed line segments joined end to end.

ii) The graph starts at the point $(-2, 3)$.

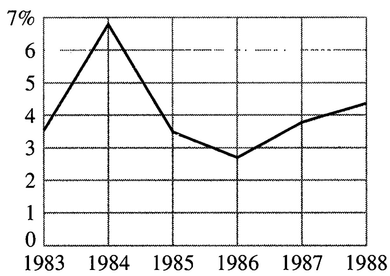
iii) The derivative of f is the step function in Fig. 2.13.



2.13 The derivative graph for Exercise 32.

b) Repeat part (a) assuming that the graph starts at $(-2, 0)$ instead of $(-2, 3)$.

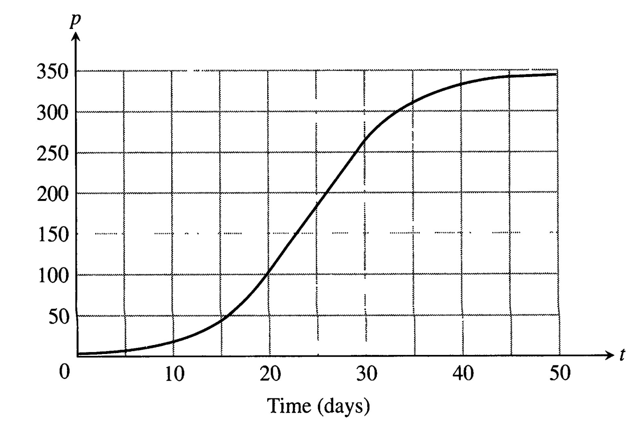
33. *Growth in the economy.* The graph in Fig. 2.14 shows the average annual percentage change $y = f(t)$ in the U.S. gross national product (GNP) for the years 1983–1988. Graph dy/dt (where defined). (Source: *Statistical Abstracts of the United States,* 110th Edition, U.S. Department of Commerce, p. 427.)



2.14 The graph for Exercise 33.

34. *Fruit flies. (Continuation of Example 3, Section 1.1.)* Populations starting out in closed environments grow slowly at first, when there are relatively few members, then more rapidly as the number of reproducing individuals increases and resources are still abundant, then slowly again as the population reaches the carrying capacity of the environment.

a) Use the graphical technique of Example 4 to graph the derivative of the fruit fly population introduced in Section 1.1. The graph of the population is reproduced here as Fig. 2.15. What units should be used on the horizontal and vertical axes for the derivative's graph?

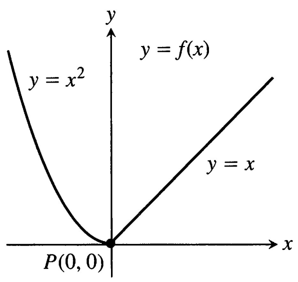


2.15 The graph for Exercise 34.

b) During what days does the population seem to be increasing fastest? slowest?

Compare the right-hand and left-hand derivatives to show that the functions in Exercises 35–38 are not differentiable at the point *P*.

35.



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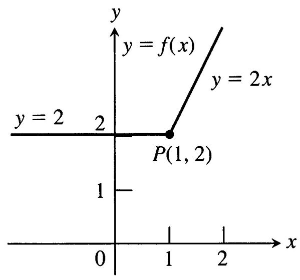

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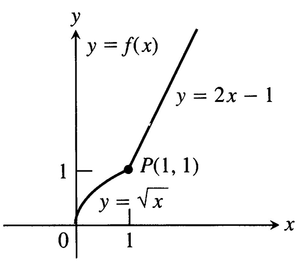

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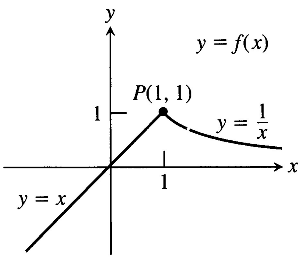
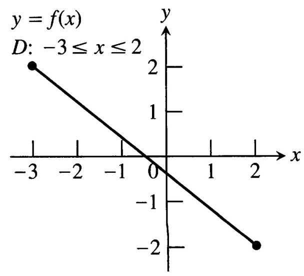


Each figure in Exercises 39–44 shows the graph of a function over a closed interval *D*. At what domain points does the function appear to be

a) differentiable?
b) continuous but not differentiable?
c) neither continuous nor differentiable?

Give reasons for your answers.

39.



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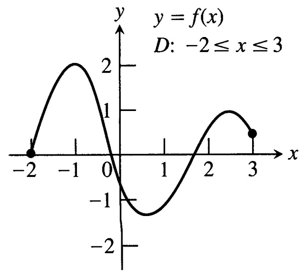

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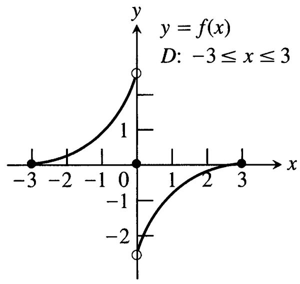

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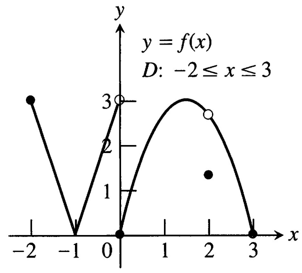

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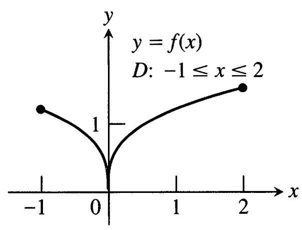

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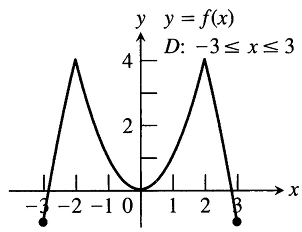


Theory and Examples

In Exercises 45–48,

a) Find the derivative $y' = f'(x)$ of the given function $y = f(x)$.
b) Graph $y = f(x)$ and $y' = f'(x)$ side by side using separate sets of coordinate axes, and answer the following questions.
c) For what values of x, if any, is y' positive? zero? negative?
d) Over what intervals of x-values, if any, does the function $y = f(x)$ increase as x increases? decrease as x increases? How is this related to what you found in (c)? (We will say more about this relationship in Chapter 3.)

45. $y = -x^2$

46. $y = -1/x$

47. $y = x^3/3$

48. $y = x^4/4$

49. Does the curve $y = x^3$ ever have a negative slope? If so, where? Give reasons for your answer.

50. Does the curve $y = 2\sqrt{x}$ have any horizontal tangents? If so, where? Give reasons for your answer.

51. Does the parabola $y = 2x^2 - 13x + 5$ have a tangent whose slope is -1? If so, find an equation for the line and the point of tangency. If not, why not?

52. Does any tangent to the curve $y = \sqrt{x}$ cross the x-axis at $x = -1$? If so, find an equation for the line and the point of tangency. If not, why not?

53. Does any function differentiable on $(-\infty, \infty)$ have $y = \lfloor x \rfloor$ as its derivative? Give reasons for your answer.

54. Graph the derivative of $f(x) = |x|$. Then graph $y = (|x| - 0)/(x - 0) = |x|/x$. What can you conclude?

55. Does knowing that a function $f(x)$ is differentiable at $x = x_0$ tell you anything about the differentiability of the function $-f$ at $x = x_0$? Give reasons for your answer.

56. Does knowing that a function $g(t)$ is differentiable at $t = 7$ tell you anything about the differentiability of the function $3g$ at $t = 7$? Give reasons for your answer.

57. Suppose that functions $g(t)$ and $h(t)$ are defined for all values of t and that $g(0) = h(0) = 0$. Can $\lim_{t\to 0}(g(t))/(h(t))$ exist? If it does exist, must it equal zero? Give reasons for your answers.

58. a) Let $f(x)$ be a function satisfying $|f(x)| \le x^2$ for $-1 \le x \le 1$. Show that f is differentiable at $x = 0$ and find $f'(0)$.

b) Show that

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is differentiable at $x = 0$ and find $f'(0)$.

Grapher Explorations

59. Graph $y = 1/(2\sqrt{x})$ in a window that has $0 \le x \le 2$. Then, on the same screen, graph

$$y = \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

for $h = 1, 0.5, 0.1$. Then try $h = -1, -0.5, -0.1$. Explain what is going on.

60. Graph $y = 3x^2$ in a window that has $-2 \le x \le 2, 0 \le y \le 3$. Then, on the same screen, graph

$$y = \frac{(x+h)^3 - x^3}{h}$$

for $h = 2, 1, 0.2$. Then try $h = -2, -1, -0.2$. Explain what is going on.

61. *Weierstrass's nowhere differentiable continuous function.* The sum of the first eight terms of the Weierstrass function $f(x) = \sum_{n=0}^{\infty}(2/3)^n \cos(9^n \pi x)$ is

$$g(x) = \cos(\pi x) + \left(\frac{2}{3}\right)^1 \cos(9\pi x) + \left(\frac{2}{3}\right)^2 \cos(9^2\pi x) + \left(\frac{2}{3}\right)^3 \cos(9^3\pi x) + \cdots + \left(\frac{2}{3}\right)^7 \cos(9^7\pi x).$$

Graph this sum. Zoom in several times. How wiggly and bumpy is this graph? Specify a viewing window in which the displayed portion of the graph is smooth.

CAS Explorations and Projects

Use a CAS to perform the following steps for the functions in Exercises 62–67.

a) Plot $y = f(x)$ to see that function's global behavior.

b) Define the difference quotient q at a general point x, with general stepsize h.

c) Take the limit as $h \to 0$. What formula does this give?

d) Substitute the value $x = x_0$ and plot the function together with its tangent line at that point.

e) Substitute various values for x larger and smaller than x_0 into the formula obtained in part (c). Do the numbers make sense with your picture?

f) Graph the formula obtained in part (c). What does it mean when its values are negative? zero? positive? Does this make sense with your plot from part (a)? Give reasons for your answer.

62. $f(x) = x^3 + x^2 - x, \quad x_0 = 1$

63. $f(x) = x^{1/3} + x^{2/3}, \quad x_0 = 1$

64. $f(x) = \dfrac{4x}{x^2+1}, \quad x_0 = 2$

65. $f(x) = \dfrac{x-1}{3x^2+1}, \quad x_0 = -1$

66. $f(x) = \sin 2x, \quad x_0 = \pi/2$

67. $f(x) = x^2 \cos x, \quad x_0 = \pi/4$

2.2 Differentiation Rules

This section shows how to differentiate functions without having to apply the definition each time.

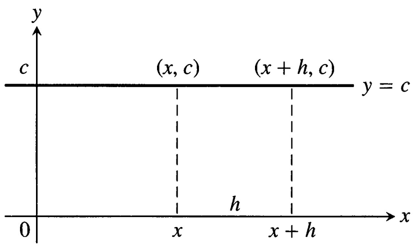
Powers, Multiples, Sums, and Differences

The first rule of differentiation is that the derivative of every constant function is zero.

Rule 1 Derivative of a Constant

If c is constant, then $\frac{d}{dx}c = 0$.

EXAMPLE 1 $\frac{d}{dx}(8) = 0, \quad \frac{d}{dx}\left(-\frac{1}{2}\right) = 0, \quad \frac{d}{dx}\left(\sqrt{3}\right) = 0$ ❑



2.16 The rule $(d/dx)(c) = 0$ is another way to say that the values of constant functions never change and that the slope of a horizontal line is zero at every point.

Proof of Rule 1 We apply the definition of derivative to $f(x) = c$, the function whose outputs have the constant value c (Fig. 2.16). At every value of x, we find that

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} 0 = 0.$$ ❑

The next rule tells how to differentiate x^n if n is a positive integer.

Rule 2 Power Rule for Positive Integers

If n is a positive integer, then

$$\frac{d}{dx}x^n = nx^{n-1}.$$

To apply the Power Rule, we subtract 1 from the original exponent (n) and multiply the result by n.

EXAMPLE 2

f	x	x^2	x^3	x^4	$\cdots$
f'	1	$2x$	$3x^2$	$4x^3$	$\cdots$

❑

Proof of Rule 2 If $f(x) = x^n$, then $f(x+h) = (x+h)^n$. Since n is a positive integer, we can use the fact that

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$$

to simplify the difference quotient for f. Taking $x + h = a$ and $x = b$, we have $a - b = h$. Thus

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{(x+h)^n - x^n}{h} \\ &= \frac{(h)[(x+h)^{n-1} + (x+h)^{n-2}x + \cdots + (x+h)x^{n-2} + x^{n-1}]}{h} \\ &= \underbrace{(x+h)^{n-1} + (x+h)^{n-2}x + \cdots + (x+h)x^{n-2} + x^{n-1}}_{n \text{ terms, each with limit } x^{n-1} \text{ as } h \to 0}.\end{aligned}$$

Hence

$$\frac{d}{dx}x^n = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = nx^{n-1}.$$ ❑

The next rule says that when a differentiable function is multiplied by a constant, its derivative is multiplied by the same constant.

Rule 3 The Constant Multiple Rule

If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

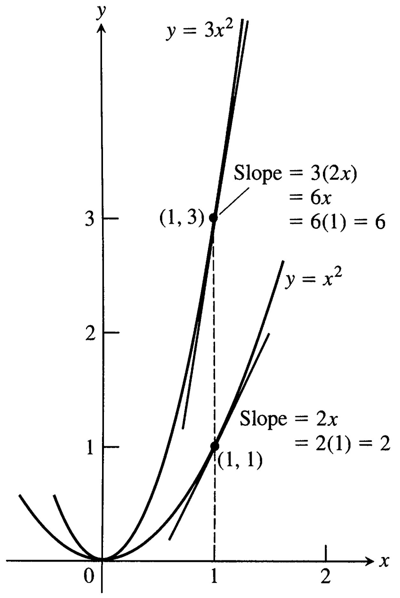
In particular, if n is a positive integer, then

$$\frac{d}{dx}(cx^n) = cn\,x^{n-1}.$$

EXAMPLE 3 The derivative formula

$$\frac{d}{dx}(3x^2) = 3 \cdot 2x = 6x$$

says that if we rescale the graph of $y = x^2$ by multiplying each y-coordinate by 3, then we multiply the slope at each point by 3 (Fig. 2.17). ❑



2.17 The graphs of $y = x^2$ and $y = 3x^2$. Tripling the y-coordinates triples the slope (Example 3).

EXAMPLE 4 *A useful special case*

The derivative of the negative of a differentiable function is the negative of the function's derivative. Rule 3 with $c = -1$ gives

$$\frac{d}{dx}(-u) = \frac{d}{dx}(-1 \cdot u) = -1 \cdot \frac{d}{dx}(u) = -\frac{du}{dx}.$$ ❑

Proof of Rule 3

$$\begin{aligned}\frac{d}{dx}cu &= \lim_{h\to 0}\frac{cu(x+h)-cu(x)}{h} && \text{Derivative definition with } f(x) = cu(x) \\ &= c\lim_{h\to 0}\frac{u(x+h)-u(x)}{h} && \text{Limit property} \\ &= c\frac{du}{dx} && u \text{ is differentiable.}\end{aligned}$$ ❑

The next rule says that the derivative of the sum of two differentiable functions is the sum of their derivatives.

Denoting functions by u and v

The functions we are working with when we need a differentiation formula are likely to be denoted by letters like f and g. When we apply the formula, we do not want to find it using these same letters in some other way. To guard against this, we denote the functions in differentiation rules by letters like u and v that are not likely to be already in use.

Rule 4 The Sum Rule

If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

Combining the Sum Rule with the Constant Multiple Rule gives the equivalent **Difference Rule,** which says that the derivative of a *difference* of differentiable functions is the difference of their derivatives.

$$\frac{d}{dx}(u - v) = \frac{d}{dx}[u + (-1)v] = \frac{du}{dx} + (-1)\frac{dv}{dx} = \frac{du}{dx} - \frac{dv}{dx}$$

The Sum Rule also extends to sums of more than two functions, as long as there are only finitely many functions in the sum. If $u_1, u_2, \ldots, u_n$ are differentiable at x, then so is $u_1 + u_2 + \cdots + u_n$, and

$$\frac{d}{dx}(u_1 + u_2 + \cdots + u_n) = \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_n}{dx}.$$

EXAMPLE 5

a) $y = x^4 + 12x$

$$\frac{dy}{dx} = \frac{d}{dx}(x^4) + \frac{d}{dx}(12x)$$

$$= 4x^3 + 12$$

b) $y = x^3 + \frac{4}{3}x^2 - 5x + 1$

$$\frac{dy}{dx} = \frac{d}{dx}x^3 + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1)$$

$$= 3x^2 + \frac{4}{3} \cdot 2x - 5 + 0$$

$$= 3x^2 + \frac{8}{3}x - 5$$

❑

Notice that we can differentiate any polynomial term by term, the way we differentiated the polynomials in Example 5.

Proof of Rule 4 We apply the definition of derivative to $f(x) = u(x) + v(x)$:

$$\frac{d}{dx}[u(x) + v(x)] = \lim_{h \to 0} \frac{[u(x+h) + v(x+h)] - [u(x) + v(x)]}{h}$$

$$= \lim_{h \to 0} \left[\frac{u(x+h) - u(x)}{h} + \frac{v(x+h) - v(x)}{h}\right]$$

$$= \lim_{h \to 0} \frac{u(x+h) - u(x)}{h} + \lim_{h \to 0} \frac{v(x+h) - v(x)}{h} = \frac{du}{dx} + \frac{dv}{dx}.$$

❑

Proof by mathematical induction

Many formulas can be shown to hold for every positive integer n greater than or equal to some lowest integer n_0 by applying an axiom called the *mathematical induction principle.* A proof using this axiom is called a *proof by mathematical induction* or a *proof by induction.* The steps in proving a formula by induction are

1. Check that it holds for $n = n_0$.
2. Prove that if it holds for any positive integer $n = k \geq n_0$, then it holds for $n = k + 1$.

Once these steps are completed, the axiom says, we know that the formula holds for all $n \geq n_0$. For more mathematical induction, see Appendix 1.

Proof of the Sum Rule for Sums of More Than Two Functions We prove the statement

$$\frac{d}{dx}(u_1 + u_2 + \cdots + u_n) = \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_n}{dx}$$

by mathematical induction. The statement is true for $n = 2$, as was just proved. This is step 1 of the induction proof.

Step 2 is to show that if the statement is true for any positive integer $n = k$, where $k \geq n_0 = 2$, then it is also true for $n = k + 1$. So suppose that

$$\frac{d}{dx}(u_1 + u_2 + \cdots + u_k) = \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_k}{dx}. \tag{1}$$

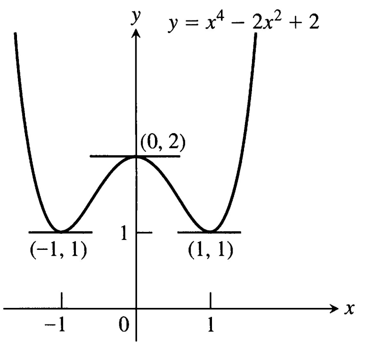
Then

$$\frac{d}{dx}\,(\underbrace{u_1 + u_2 + \cdots + u_k}_{\substack{\text{Call the function}\\ \text{defined by this sum } u.}} + \underbrace{u_{k+1}}_{\substack{\text{Call this}\\ \text{function } v.}})$$

$$= \frac{d}{dx}(u_1 + u_2 + \cdots + u_k) + \frac{du_{k+1}}{dx} \qquad \text{Rule 4 for } \frac{d}{dx}(u + v)$$

$$= \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_k}{dx} + \frac{du_{k+1}}{dx}. \qquad \text{Eq. (1)}$$

With these steps verified, the mathematical induction principle now guarantees the Sum Rule for every integer $n \geq 2$. ❑



2.18 The curve $y = x^4 - 2x^2 + 2$ and its horizontal tangents (Example 6).

EXAMPLE 6 Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so, where?

Solution The horizontal tangents, if any, occur where the slope dy/dx is zero. To find these points, we

1. Calculate dy/dx: $\quad \dfrac{dy}{dx} = \dfrac{d}{dx}(x^4 - 2x^2 + 2) = 4x^3 - 4x$
2. Solve the equation $\dfrac{dy}{dx} = 0$ for x: $\quad 4x^3 - 4x = 0$

$$4x(x^2 - 1) = 0$$
$$x = 0, 1, -1$$

The curve $y = x^4 - 2x^2 + 2$ has horizontal tangents at $x = 0, 1,$ and -1. The corresponding points on the curve are (0, 2), (1, 1) and (−1, 1). See Fig. 2.18. ❑

Products and Quotients

While the derivative of the sum of two functions is the sum of their derivatives, the derivative of the product of two functions is *not* the product of their derivatives. For instance,

$$\frac{d}{dx}(x \cdot x) = \frac{d}{dx}(x^2) = 2x, \qquad \text{while} \qquad \frac{d}{dx}(x) \cdot \frac{d}{dx}(x) = 1 \cdot 1 = 1.$$

The derivative of a product of two functions is the sum of *two* products, as we now explain.

Rule 5 The Product Rule

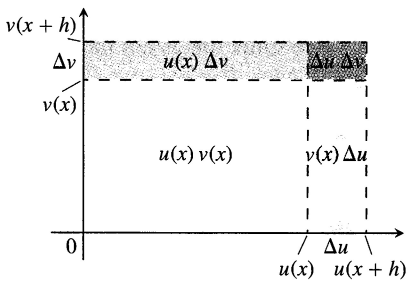
If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

The derivative of the product uv is u times the derivative of v plus v times the derivative of u. In *prime notation,* $(uv)' = uv' + vu'$.

Picturing the product rule

If $u(x)$ and $v(x)$ are positive and increase when x increases, and if $h > 0$,



the total shaded area in the picture is

$$u(x+h)v(x+h) - u(x)v(x) =$$
$$u(x+h)\Delta v + v(x+h)\Delta u - \Delta u \Delta v.$$

Dividing both sides of this equation by h gives

$$\frac{u(x+h)v(x+h) - u(x)v(x)}{h}$$
$$= u(x+h)\frac{\Delta v}{h} + v(x+h)\frac{\Delta u}{h} - \Delta u \frac{\Delta v}{h}.$$

As $h \to 0^+$, $\Delta u \cdot \dfrac{\Delta v}{h} \to 0 \cdot \dfrac{dv}{dx} = 0$, leaving

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Proof of Rule 5

$$\frac{d}{dx}(uv) = \lim_{h\to 0} \frac{u(x+h)v(x+h) - u(x)v(x)}{h}$$

To change this fraction into an equivalent one that contains difference quotients for the derivatives of u and v, we subtract and add $u(x+h)v(x)$ in the numerator:

$$\begin{aligned}\frac{d}{dx}(uv) &= \lim_{h\to 0} \frac{u(x+h)v(x+h) - u(x+h)v(x) + u(x+h)v(x) - u(x)v(x)}{h} \\ &= \lim_{h\to 0}\left[u(x+h)\frac{v(x+h)-v(x)}{h} + v(x)\frac{u(x+h)-u(x)}{h}\right] \\ &= \lim_{h\to 0} u(x+h) \cdot \lim_{h\to 0}\frac{v(x+h)-v(x)}{h} + v(x)\cdot \lim_{h\to 0}\frac{u(x+h)-u(x)}{h}.\end{aligned}$$

As h approaches zero, $u(x+h)$ approaches $u(x)$ because u, being differentiable at x, is continuous at x. The two fractions approach the values of dv/dx at x and du/dx at x. In short,

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$ ❑

EXAMPLE 7 Find the derivative of $y = (x^2+1)(x^3+3)$.

Solution From the Product Rule with $u = x^2 + 1$ and $v = x^3 + 3$, we find

$$\begin{aligned}\frac{d}{dx}[(x^2+1)(x^3+3)] &= (x^2+1)(3x^2) + (x^3+3)(2x) \\ &= 3x^4 + 3x^2 + 2x^4 + 6x \\ &= 5x^4 + 3x^2 + 6x.\end{aligned}$$ ❑

Example 7 can be done as well (perhaps better) by multiplying out the original expression for y and differentiating the resulting polynomial. We now check:

$$y = (x^2+1)(x^3+3) = x^5 + x^3 + 3x^2 + 3$$
$$\frac{dy}{dx} = 5x^4 + 3x^2 + 6x.$$

This is in agreement with our first calculation.

There are times, however, when the Product Rule *must* be used. In the following example, we have only numerical values to work with.

EXAMPLE 8 Let $y = uv$ be the product of the functions u and v. Find $y'(2)$ if

$$u(2) = 3, \quad u'(2) = -4, \quad v(2) = 1, \quad \text{and} \quad v'(2) = 2.$$

Solution From the Product Rule, in the form

$$y' = (uv)' = uv' + vu',$$

we have

$$\begin{aligned} y'(2) &= u(2)v'(2) + v(2)u'(2) \\ &= (3)(2) + (1)(-4) = 6 - 4 = 2. \end{aligned}$$

❑

Quotients

Just as the derivative of the product of two differentiable functions is not the product of their derivatives, the derivative of the quotient of two functions is not the quotient of their derivatives. What happens instead is this:

> **Rule 6 The Quotient Rule**
>
> If u and v are differentiable at x, and $v(x) \neq 0$, then the quotient u/v is differentiable at x, and
>
> $$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

Proof of Rule 6

$$\begin{aligned} \frac{d}{dx}\left(\frac{u}{v}\right) &= \lim_{h\to 0} \frac{\dfrac{u(x+h)}{v(x+h)} - \dfrac{u(x)}{v(x)}}{h} \\ &= \lim_{h\to 0} \frac{v(x)u(x+h) - u(x)v(x+h)}{hv(x+h)v(x)} \end{aligned}$$

To change the last fraction into an equivalent one that contains the difference quotients for the derivatives of u and v, we subtract and add $v(x)u(x)$ in the numerator. We then get

$$\begin{aligned} \frac{d}{dx}\left(\frac{u}{v}\right) &= \lim_{h\to 0} \frac{v(x)u(x+h) - v(x)u(x) + v(x)u(x) - u(x)v(x+h)}{hv(x+h)v(x)} \\ &= \lim_{h\to 0} \frac{v(x)\dfrac{u(x+h) - u(x)}{h} - u(x)\dfrac{v(x+h) - v(x)}{h}}{v(x+h)v(x)}. \end{aligned}$$

Taking the limit in the numerator and denominator now gives the Quotient Rule.

❑

EXAMPLE 9 Find the derivative of $y = \dfrac{t^2 - 1}{t^2 + 1}$.

Solution We apply the Quotient Rule with $u = t^2 - 1$ and $v = t^2 + 1$:

$$\frac{dy}{dt} = \frac{(t^2+1)\cdot 2t - (t^2-1)\cdot 2t}{(t^2+1)^2} \qquad \frac{d}{dt}\left(\frac{u}{v}\right) = \frac{v(du/dt) - u(dv/dt)}{v^2}$$

$$= \frac{2t^3 + 2t - 2t^3 + 2t}{(t^2+1)^2}$$

$$= \frac{4t}{(t^2+1)^2}.$$

❑

The Power Rule for Negative Integers

The Power Rule for negative integers is the same as the rule for positive integers.

> **Rule 7 Power Rule for Negative Integers**
>
> If n is a negative integer and $x \neq 0$, then
>
> $$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Proof of Rule 7 The proof uses the Quotient Rule in a clever way. If n is a negative integer, then $n = -m$ where m is a positive integer. Hence, $x^n = x^{-m} = 1/x^m$ and

$$\frac{d}{dx}(x^n) = \frac{d}{dx}\left(\frac{1}{x^m}\right)$$

$$= \frac{x^m \cdot \frac{d}{dx}(1) - 1 \cdot \frac{d}{dx}(x^m)}{(x^m)^2} \qquad \text{Quotient Rule with } u = 1 \text{ and } v = x^m$$

$$= \frac{0 - mx^{m-1}}{x^{2m}} \qquad \text{Since } m > 0,\ \frac{d}{dx}(x^m) = mx^{m-1}$$

$$= -mx^{-m-1}$$

$$= nx^{n-1}. \qquad \text{Since } -m = n$$

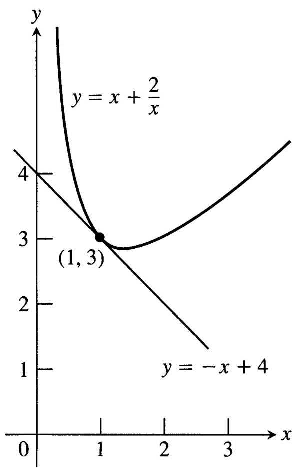
❑

EXAMPLE 10

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = (-1)x^{-2} = -\frac{1}{x^2}$$

$$\frac{d}{dx}\left(\frac{4}{x^3}\right) = 4\frac{d}{dx}(x^{-3}) = 4(-3)x^{-4} = -\frac{12}{x^4}$$

❑



2.19 The tangent to the curve $y = x + (2/x)$ at (1, 3). The curve has a third-quadrant portion not shown here. We will see how to graph functions like this in Chapter 3.

EXAMPLE 11 Find an equation for the tangent to the curve

$$y = x + \frac{2}{x}$$

at the point (1, 3) (Fig. 2.19).

Solution The slope of the curve is

$$\frac{dy}{dx} = \frac{d}{dx}(x) + 2\frac{d}{dx}\left(\frac{1}{x}\right) = 1 + 2\left(-\frac{1}{x^2}\right) = 1 - \frac{2}{x^2}.$$

The slope at $x = 1$ is

$$\left.\frac{dy}{dx}\right|_{x=1} = \left[1 - \frac{2}{x^2}\right]_{x=1} = 1 - 2 = -1.$$

The line through (1, 3) with slope $m = -1$ is

$$y - 3 = (-1)(x - 1) \qquad \text{Point-slope equation}$$
$$y = -x + 1 + 3$$
$$y = -x + 4.$$

❑

Choosing Which Rules to Use

The choice of which rules to use in solving a differentiation problem can make a difference in how much work you have to do. Here is an example.

EXAMPLE 12 Rather than using the Quotient Rule to find the derivative of

$$y = \frac{(x-1)(x^2-2x)}{x^4},$$

expand the numerator and divide by x^4:

$$y = \frac{(x-1)(x^2-2x)}{x^4} = \frac{x^3 - 3x^2 + 2x}{x^4} = x^{-1} - 3x^{-2} + 2x^{-3}.$$

Then use the Sum and Power Rules:

$$\frac{dy}{dx} = -x^{-2} - 3(-2)x^{-3} + 2(-3)x^{-4}$$
$$= -\frac{1}{x^2} + \frac{6}{x^3} - \frac{6}{x^4}.$$

❑

Second and Higher Order Derivatives

The derivative $y' = dy/dx$ is the **first (first order) derivative** of y with respect to x. This derivative may itself be a differentiable function of x; if so, its derivative

$$y'' = \frac{dy'}{dx} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

is called the **second (second order) derivative** of y with respect to x.

If y'' is differentiable, its derivative, $y''' = dy''/dx = d^3y/dx^3$ is the **third (third order) derivative** of y with respect to x. The names continue as you imagine, with

$$y^{(n)} = \frac{d}{dx}y^{(n-1)}$$

denoting the ***n*th (*n*th order) derivative** of y with respect to x, for any positive integer n.

Notice that

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)$$

does not mean multiplication. It means "the derivative of the derivative."
