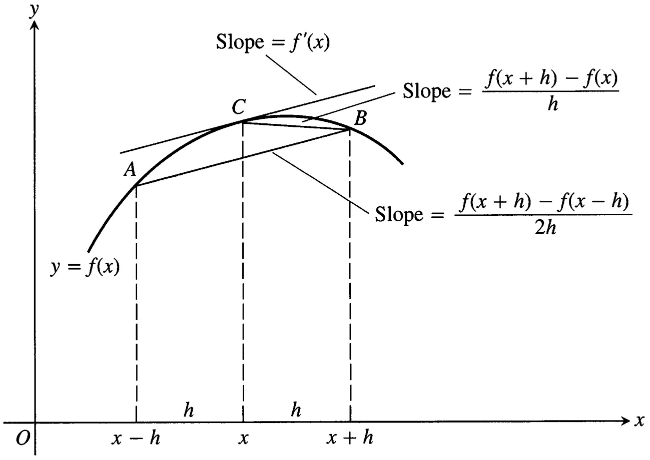
See the figure below.



a) To see how rapidly the centered difference quotient for $f(x) = \sin x$ converges to $f'(x) = \cos x$, graph $y = \cos x$ together with

$$y = \frac{\sin(x+h) - \sin(x-h)}{2h}$$

over the interval $[-\pi, 2\pi]$ for $h = 1, 0.5$, and 0.3. Compare the results with those obtained in Exercise 69 for the same values of h.

b) To see how rapidly the centered difference quotient for $f(x) = \cos x$ converges to $f'(x) = -\sin x$, graph $y = -\sin x$ together with

$$y = \frac{\cos(x+h) - \cos(x-h)}{2h}$$

over the interval $[-\pi, 2\pi]$ for $h = 1, 0.5$, and 0.3. Compare the results with those obtained in Exercise 70 for the same values of h.

72. *A caution about centered difference quotients. (Continuation of Exercise 71.)* The quotient

$$\frac{f(x+h) - f(x-h)}{2h}$$

may have a limit as $h \to 0$ when f has no derivative at x. As a case in point, take $f(x) = |x|$ and calculate

$$\lim_{h \to 0} \frac{|0+h| - |0-h|}{2h}.$$

As you will see, the limit exists even though $f(x) = |x|$ has no derivative at $x = 0$.

73. Graph $y = \tan x$ and its derivative together on $(-\pi/2, \pi/2)$. Does the graph of the tangent function appear to have a smallest slope? a largest slope? Is the slope ever negative? Give reasons for your answers.

74. Graph $y = \cot x$ and its derivative together for $0 < x < \pi$. Does the graph of the cotangent function appear to have a smallest slope? a largest slope? Is the slope ever positive? Give reasons for your answers.

75. Graph $y = (\sin x)/x$, $y = (\sin 2x)/x$, and $y = (\sin 4x)/x$ together over the interval $-2 \le x \le 2$. Where does each graph appear to cross the y-axis? Do the graphs really intersect the axis? What would you expect the graphs of $y = (\sin 5x)/x$ and $y = (\sin(-3x))/x$ to do as $x \to 0$? Why? What about the graph of $y = (\sin kx)/x$ for other values of k? Give reasons for your answers.

76. *Radians vs. degrees.* What happens to the derivatives of sin x and cos x if x is measured in degrees instead of radians? To find out, take the following steps.

a) With your graphing calculator or computer grapher *in degree mode,* graph

$$f(h) = \frac{\sin h}{h}$$

and estimate $\lim_{h \to 0} f(h)$. Compare your estimate with $\pi/180$. Is there any reason to believe the limit *should* be $\pi/180$?

b) With your grapher still in degree mode, estimate

$$\lim_{h \to 0} \frac{\cos h - 1}{h}.$$

c) Now go back to the derivation of the formula for the derivative of sin x in the text and carry out the steps of the derivation using degree-mode limits. What formula do you obtain for the derivative?

d) Work through the derivation of the formula for the derivative of cos x using degree-mode limits. What formula do you obtain for the derivative?

e) The disadvantages of the degree-mode formulas become apparent as you start taking derivatives of higher order. Try it. What are the second and third degree-mode derivatives of sin x and cos x?

2.5 The Chain Rule

We now know how to differentiate $\sin x$ and $x^2 - 4$, but how do we differentiate a composite like $\sin(x^2 - 4)$? The answer is, with the Chain Rule, which says that the derivative of the composite of two differentiable functions is the product of their derivatives evaluated at appropriate points. The Chain Rule is probably the most widely used differentiation rule in mathematics. This section describes the rule and how to use it. We begin with examples.

EXAMPLE 1 The function $y = 6x - 10 = 2(3x - 5)$ is the composite of the functions $y = 2u$ and $u = 3x - 5$. How are the derivatives of these three functions related?

Solution We have

$$\frac{dy}{dx} = 6, \qquad \frac{dy}{du} = 2, \qquad \frac{du}{dx} = 3.$$

Since $6 = 2 \cdot 3$,

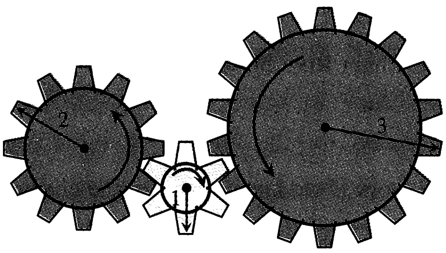
$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}. \qquad \square$$

Is it an accident that

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}?$$

If we think of the derivative as a rate of change, our intuition allows us to see that this relationship is reasonable. For $y = f(u)$ and $u = g(x)$, if y changes twice as fast as u and u changes three times as fast as x, then we expect y to change six times as fast as x. This is much like the effect of a multiple gear train (Fig. 2.39).

Let us try this again on another function.



2.39 When gear A makes x turns, gear B makes u turns and gear C makes y turns. By comparing circumferences or counting teeth, we see that $y = u/2$ and $u = 3x$, so $y = 3x/2$. Thus $dy/du = 1/2$, $du/dx = 3$, and $dy/dx = 3/2 = (dy/du)(du/dx)$.

EXAMPLE 2

$$y = 9x^4 + 6x^2 + 1 = (3x^2 + 1)^2$$

is the composite of $y = u^2$ and $u = 3x^2 + 1$. Calculating derivatives, we see that

$$\begin{aligned} \frac{dy}{du} \cdot \frac{du}{dx} &= 2u \cdot 6x \\ &= 2(3x^2 + 1) \cdot 6x \\ &= 36x^3 + 12x \end{aligned}$$

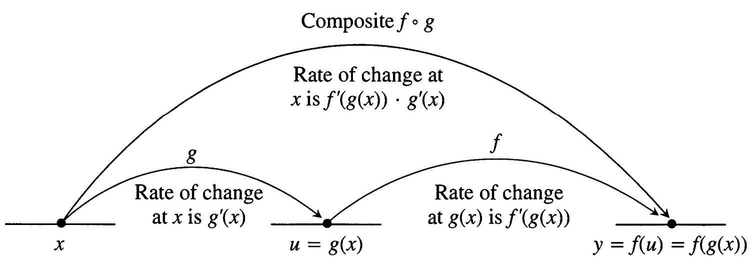
and

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(9x^4 + 6x^2 + 1) \\ &= 36x^3 + 12x. \end{aligned}$$

Once again,

$$\frac{dy}{du} \cdot \frac{du}{dx} = \frac{dy}{dx}. \qquad \square$$

The derivative of the composite function $f(g(x))$ at x is the derivative of f at $g(x)$ times the derivative of g at x. This is known as the Chain Rule (Fig. 2.40).



2.40 Rates of change multiply: the derivative of $f \circ g$ at x is the derivative of f at the point $g(x)$ times the derivative of g at x.

Theorem 5
The Chain Rule
If $f(u)$ is differentiable at the point $u = g(x)$, and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x). \tag{1}$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}, \tag{2}$$

where dy/du is evaluated at $u = g(x)$.

It would be tempting to try to prove the Chain Rule by writing

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$$

and taking the limit as $\Delta x \to 0$. This would work if we knew that Δu, the change in u, was nonzero, but we do not know this. A small change in x could conceivably produce no change in u. The proof requires a different approach, using ideas in Section 3.7. We will return to it when the time comes.

EXAMPLE 3 Find the derivative of $y = \sqrt{x^2 + 1}$.

Solution Here $y = f(g(x))$, where $f(u) = \sqrt{u}$ and $g(x) = x^2 + 1$. Since the derivatives of f and g are

$$f'(u) = \frac{1}{2\sqrt{u}} \quad \text{and} \quad g'(x) = 2x,$$

the Chain Rule gives

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x) \\ &= \frac{1}{2\sqrt{g(x)}} \cdot g'(x) = \frac{1}{2\sqrt{x^2+1}} \cdot (2x) \\ &= \frac{x}{\sqrt{x^2+1}}.\end{aligned}$$

❑

The "Outside-Inside" Rule

It sometimes helps to think about the Chain Rule the following way. If $y = f(g(x))$, Eq. (2) tells us that

$$\frac{dy}{dx} = f'[g(x)] \cdot g'(x). \tag{3}$$

In words, Eq. (3) says: To find dy/dx, differentiate the "outside" function f and leave the "inside" $g(x)$ alone; then multiply by the derivative of the inside.

EXAMPLE 4

$$\frac{d}{dx}\overset{\text{outside}}{\sin}\underbrace{(x^2+x)}_{\text{inside}} = \overset{\text{derivative of the outside}}{\cos}\underbrace{(x^2+x)}_{\text{inside left alone}} \cdot \underbrace{(2x+1)}_{\text{derivative of the inside}}$$

❑

Repeated Use of the Chain Rule

We sometimes have to use the Chain Rule two or more times to find a derivative. Here is an example.

EXAMPLE 5 Find the derivative of $g(t) = \tan(5 - \sin 2t)$.

Solution

$$\begin{aligned}
g'(t) &= \frac{d}{dt}(\tan(5 - \sin 2t)) \\
&= \sec^2(5 - \sin 2t) \cdot \frac{d}{dt}(5 - \sin 2t) && \text{Derivative of } \tan u \text{ with } u = 5 - \sin 2t \\
&= \sec^2(5 - \sin 2t) \cdot \left(0 - (\cos 2t) \cdot \frac{d}{dt}(2t)\right) && \text{Derivative of } 5 - \sin u \text{ with } u = 2t \\
&= \sec^2(5 - \sin 2t) \cdot (-\cos 2t) \cdot 2 \\
&= -2(\cos 2t)\sec^2(5 - \sin 2t)
\end{aligned}$$

❑

Differentiation Formulas That Include the Chain Rule

Many of the differentiation formulas you will encounter in your scientific work already include the Chain Rule.

If f is a differentiable function of u, and u is a differentiable function of x, then substituting $y = f(u)$ in the Chain Rule formula

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

leads to the formula

$$\frac{d}{dx}f(u) = f'(u)\frac{du}{dx}. \tag{4}$$

For example, if u is a differentiable function of x, n is an integer, and $y = u^n$, then the Chain Rule gives

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{du}(u^n) \cdot \frac{du}{dx} \\
&= nu^{n-1}\frac{du}{dx}. && \text{Differentiating } u^n \text{ with respect to } u \text{ itself gives } nu^{n-1}.
\end{aligned}$$

Power Chain Rule

If $u(x)$ is a differentiable function and n is an integer, then u^n is differentiable and

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}. \tag{5}$$

$\sin^n x$ is short for $(\sin x)^n$, $n \neq -1$.

EXAMPLE 6

a) $\frac{d}{dx}\sin^5 x = 5\sin^4 x \frac{d}{dx}(\sin x)$ Eq. (5) with $u = \sin x$, $n = 5$

$= 5\sin^4 x \cos x$

b) $\frac{d}{dx}(2x+1)^{-3} = -3(2x+1)^{-4}\frac{d}{dx}(2x+1)$ Eq. (5) with $u = 2x+1$, $n = -3$

$= -3(2x+1)^{-4}(2)$

$= -6(2x+1)^{-4}$

c) $\frac{d}{dx}(5x^3 - x^4)^7 = 7(5x^3 - x^4)^6\frac{d}{dx}(5x^3 - x^4)$ Eq. (5) with $u = 5x^3 - x^4$, $n = 7$

$= 7(5x^3 - x^4)^6(5 \cdot 3x^2 - 4x^3)$

$= 7(5x^3 - x^4)^6(15x^2 - 4x^3)$

d) $\frac{d}{dx}\left(\frac{1}{3x-2}\right) = \frac{d}{dx}(3x-2)^{-1}$ Eq. (5) with $u = 3x - 2$, $n = -1$

$= -1(3x-2)^{-2}\frac{d}{dx}(3x-2)$

$= -1(3x-2)^{-2}(3)$

$= -\frac{3}{(3x-2)^2}$

In part (d) we could also have found the derivative with the Quotient Rule. ❑

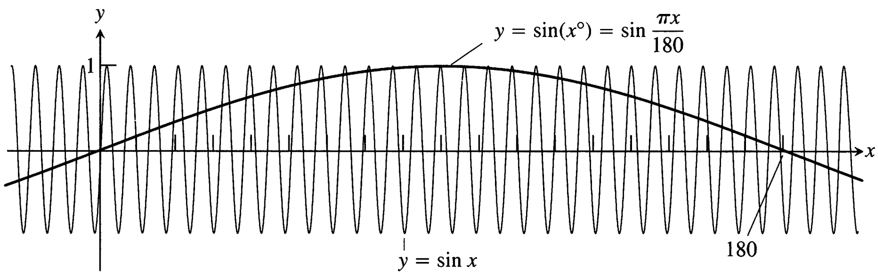
EXAMPLE 7 *Radians vs. degrees*

It is important to remember that the formulas for the derivatives of $\sin x$ and $\cos x$ were obtained under the assumption that x is measured in radians, *not* degrees. The Chain Rule brings new understanding to the difference between the two. Since $180° = \pi$ radians, $x° = \pi x/180$ radians. By the Chain Rule,

$$\frac{d}{dx}\sin(x°) = \frac{d}{dx}\sin\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos(x°).$$

See Fig. 2.41. Similarly, the derivative of $\cos(x°)$ is $-(\pi/180)\sin(x°)$.

The factor $\pi/180$, annoying in the first derivative, would compound with repeated differentiation. We see at a glance the compelling reason for the use of radian measure. ❑



2.41 $\sin(x°)$ oscillates only $\pi/180$ times as often as $\sin x$ oscillates. Its maximum slope is $\pi/180$.

Melting Ice Cubes

In mathematics, we tend to use letters like f, g, x, y, and u for functions and variables. However, other fields use letters like V, for volume, and s, for side, that come from the names of the things being modeled. The letters in the Chain Rule then change too, as in the next example.

EXAMPLE 8 *The melting ice cube*

How long will it take an ice cube to melt?

Solution As with all applications to science, we start with a mathematical model. We assume that the cube retains its cubical shape as it melts. We call its side length s, so its volume is $V = s^3$. We assume that V and s are differentiable functions of time t. We assume also that the cube's volume decreases at a rate that is proportional to its surface area. This latter assumption seems reasonable enough when we think that the melting takes place at the surface: Changing the amount of surface changes the amount of ice exposed to melt. In mathematical terms,

$$\frac{dV}{dt} = -k(6s^2), \qquad k > 0.$$

The minus sign indicates that the volume is decreasing. We assume that the proportionality factor k is constant. (It probably depends on many things, however, such as the relative humidity of the surrounding air, the air temperature, and the incidence or absence of sunlight, to name only a few.)

Finally, we need at least one more piece of information: How long will it take a specific percentage of the ice cube to melt? We have nothing to guide us unless we make one or more observations, but now let us assume a particular set of conditions in which the cube lost 1/4 of its volume during the first hour. (You could use letters instead of particular numbers: say $n\%$ in r hours. Then your answer would be in terms of n and r.)

Mathematically, we now have the following problem.

Given: $V = s^3$ and $\dfrac{dV}{dt} = -k(6s^2)$

$V = V_0$ when $t = 0$

$V = (3/4)V_0$ when $t = 1$ h

Find: The value of t when $V = 0$

We apply the Chain Rule to differentiate $V = s^3$ with respect to t:

$$\frac{dV}{dt} = 3s^2 \frac{ds}{dt}.$$

We set this equal to the given rate, $-k(6s^2)$, to get

$$3s^2 \frac{ds}{dt} = -6ks^2$$
$$\frac{ds}{dt} = -2k.$$

The side length is *decreasing* at the constant rate of $2k$ units per hour. Thus, if the initial length of the cube's side is s_0, the length of its side one hour later is $s_1 = s_0 - 2k$. This equation tells us that

$$2k = s_0 - s_1.$$

The melting time is the value of t that makes $2kt = s_0$. Hence,

$$t_{\text{melt}} = \frac{s_0}{2k} = \frac{s_0}{s_0 - s_1} = \frac{1}{1 - (s_1/s_0)}.$$

But

$$\frac{s_1}{s_0} = \frac{\left(\frac{3}{4}V_0\right)^{1/3}}{(V_0)^{1/3}} = \left(\frac{3}{4}\right)^{1/3} \approx 0.91.$$

Therefore,

$$t_{\text{melt}} = \frac{1}{1 - 0.91} \approx 11 \text{ h}.$$

If 1/4 of the cube melts in 1 h, it will take about 10 h more for the rest of it to melt. ❑

If we were natural scientists interested in testing the assumptions on which our mathematical model is based, our next step would be to run a number of experiments and compare their outcomes with the model's predictions. One practical application might lie in analyzing the proposal to tow large icebergs from polar waters to offshore locations near southern California, where the melting ice could provide fresh water. As a first approximation, we might imagine the iceberg to be a large cube or rectangular solid, or perhaps a pyramid. We will say more about mathematical modeling in Section 4.2.

Exercises 2.5

Derivative Calculations

In Exercises 1–8, given $y = f(u)$ and $u = g(x)$, find $dy/dx = f'(g(x))g'(x)$.

1. $y = 6u - 9, \quad u = (1/2)x^4$

2. $y = 2u^3, \quad u = 8x - 1$

3. $y = \sin u, \quad u = 3x + 1$

4. $y = \cos u, \quad u = -x/3$

5. $y = \cos u, \quad u = \sin x$

6. $y = \sin u, \quad u = x - \cos x$

7. $y = \tan u, \quad u = 10x - 5$

8. $y = -\sec u, \quad u = x^2 + 7x$

In Exercises 9–18, write the function in the form $y = f(u)$ and $u = g(x)$. Then find dy/dx as a function of x.

9. $y = (2x+1)^5$

10. $y = (4-3x)^9$

11. $y = \left(1 - \frac{x}{7}\right)^{-7}$

12. $y = \left(\frac{x}{2} - 1\right)^{-10}$

13. $y = \left(\frac{x^2}{8} + x - \frac{1}{x}\right)^4$

14. $y = \left(\frac{x}{5} + \frac{1}{5x}\right)^5$

15. $y = \sec(\tan x)$

16. $y = \cot\left(\pi - \frac{1}{x}\right)$

17. $y = \sin^3 x$

18. $y = 5\cos^{-4} x$

Find the derivatives of the functions in Exercises 19–38.

19. $p = \sqrt{3-t}$

20. $q = \sqrt{2r - r^2}$

21. $s = \frac{4}{3\pi}\sin 3t + \frac{4}{5\pi}\cos 5t$

22. $s = \sin\left(\frac{3\pi t}{2}\right) + \cos\left(\frac{3\pi t}{2}\right)$

23. $r = (\csc\theta + \cot\theta)^{-1}$

24. $r = -(\sec\theta + \tan\theta)^{-1}$

25. $y = x^2\sin^4 x + x\cos^{-2} x$

26. $y = \frac{1}{x}\sin^{-5} x - \frac{x}{3}\cos^3 x$

27. $y = \frac{1}{21}(3x-2)^7 + \left(4 - \frac{1}{2x^2}\right)^{-1}$

28. $y = (5-2x)^{-3} + \frac{1}{8}\left(\frac{2}{x} + 1\right)^4$

29. $y = (4x+3)^4(x+1)^{-3}$

30. $y = (2x-5)^{-1}(x^2-5x)^6$

31. $h(x) = x\tan(2\sqrt{x}) + 7$

32. $k(x) = x^2\sec\left(\frac{1}{x}\right)$

33. $f(\theta) = \left(\frac{\sin\theta}{1+\cos\theta}\right)^2$

34. $g(t) = \left(\frac{1+\cos t}{\sin t}\right)^{-1}$

35. $r = \sin(\theta^2)\cos(2\theta)$

36. $r = \sec\sqrt{\theta}\tan\left(\frac{1}{\theta}\right)$

37. $q = \sin\left(\frac{t}{\sqrt{t+1}}\right)$

38. $q = \cot\left(\frac{\sin t}{t}\right)$

In Exercises 39–48, find dy/dt.

39. $y = \sin^2(\pi t - 2)$

40. $y = \sec^2 \pi t$

41. $y = (1 + \cos 2t)^{-4}$

42. $y = (1 + \cot(t/2))^{-2}$

43. $y = \sin(\cos(2t-5))$

44. $y = \cos\left(5\sin\left(\frac{t}{3}\right)\right)$

45. $y = \left(1 + \tan^4\left(\frac{t}{12}\right)\right)^3$

46. $y = \frac{1}{6}\left(1 + \cos^2(7t)\right)^3$

47. $y = \sqrt{1 + \cos(t^2)}$

48. $y = 4\sin\left(\sqrt{1+\sqrt{t}}\right)$

Find y'' in Exercises 49–52.

49. $y = \left(1 + \frac{1}{x}\right)^3$

50. $y = (1 - \sqrt{x})^{-1}$

51. $y = \frac{1}{9}\cot(3x-1)$

52. $y = 9\tan\left(\frac{x}{3}\right)$

Finding Numerical Values of Derivatives

In Exercises 53–58, find the value of $(f \circ g)'$ at the given value of x.

53. $f(u) = u^5 + 1, \quad u = g(x) = \sqrt{x}, \quad x = 1$

54. $f(u) = 1 - \frac{1}{u}, \quad u = g(x) = \frac{1}{1-x}, \quad x = -1$

55. $f(u) = \cot\frac{\pi u}{10}, \quad u = g(x) = 5\sqrt{x}, \quad x = 1$

56. $f(u) = u + \frac{1}{\cos^2 u}, \quad u = g(x) = \pi x, \quad x = 1/4$

57. $f(u) = \frac{2u}{u^2+1}, \quad u = g(x) = 10x^2 + x + 1, \quad x = 0$

58. $f(u) = \left(\frac{u-1}{u+1}\right)^2, \quad u = g(x) = \frac{1}{x^2} - 1, \quad x = -1$

59. Suppose that functions f and g and their derivatives with respect to x have the following values at $x = 2$ and $x = 3$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	8	2	1/3	−3
3	3	−4	2π	5

Find the derivatives with respect to x of the following combinations at the given value of x.

a) $2f(x), \quad x = 2$

b) $f(x) + g(x), \quad x = 3$

c) $f(x) \cdot g(x), \quad x = 3$

d) $f(x)/g(x), \quad x = 2$

e) $f(g(x)), \quad x = 2$

f) $\sqrt{f(x)}, \quad x = 2$

g) $1/g^2(x), \quad x = 3$

h) $\sqrt{f^2(x) + g^2(x)}, \quad x = 2$

60. Suppose that the functions f and g and their derivatives with respect to x have the following values at $x = 0$ and $x = 1$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
0	1	1	5	1/3
1	3	−4	−1/3	−8/3

Find the derivatives with respect to x of the following combinations at the given value of x,

a) $5f(x) - g(x), \quad x = 1$

b) $f(x)g^3(x), \quad x = 0$

c) $\frac{f(x)}{g(x)+1}, \quad x = 1$

d) $f(g(x)), \quad x = 0$

e) $g(f(x)), \quad x = 0$

f) $(x^{11} + f(x))^{-2}, \quad x = 1$

g) $f(x + g(x)), \quad x = 0$

61. Find ds/dt when $\theta = 3\pi/2$ if $s = \cos\theta$ and $d\theta/dt = 5$.

62. Find dy/dt when $x = 1$ if $y = x^2 + 7x - 5$ and $dx/dt = 1/3$.

Choices in Composition

What happens if you can write a function as a composite in different ways? Do you get the same derivative each time? The Chain Rule says you should. Try it with the functions in Exercises 63 and 64.

63. Find dy/dx if $y = x$ by using the Chain Rule with y as a composite of

a) $y = (u/5) + 7$ and $u = 5x - 35$
b) $y = 1 + (1/u)$ and $u = 1/(x - 1)$.

64. Find dy/dx if $y = x^{3/2}$ by using the Chain Rule with y as a composite of

a) $y = u^3$ and $u = \sqrt{x}$
b) $y = \sqrt{u}$ and $u = x^3$.

Tangents and Slopes

65. a) Find the tangent to the curve $y = 2\tan(\pi x/4)$ at $x = 1$.
b) What is the smallest value the slope of the curve can ever have on the interval $-2 < x < 2$? Give reasons for your answer.

66. a) Find equations for the tangents to the curves $y = \sin 2x$ and $y = -\sin(x/2)$ at the origin. Is there anything special about how the tangents are related? Give reasons for your answer.
b) Can anything be said about the tangents to the curves $y = \sin mx$ and $y = -\sin(x/m)$ at the origin (m a constant $\neq 0$)? Give reasons for your answer.
c) For a given m, what are the largest values the slopes of the curves $y = \sin mx$ and $y = -\sin(x/m)$ can ever have? Give reasons for your answer.
d) The function $y = \sin x$ completes one period on the interval $[0, 2\pi]$, the function $y = \sin 2x$ completes two periods, the function $y = \sin(x/2)$ completes half a period, and so on. Is there any relation between the number of periods $y = \sin mx$ completes on $[0, 2\pi]$ and the slope of the curve $y = \sin mx$ at the origin? Give reasons for your answer.

Theory, Examples, and Applications

67. *Running machinery too fast.* Suppose that a piston is moving straight up and down and that its position at time t seconds is

$$s = A\cos(2\pi bt),$$

with A and b positive. The value of A is the amplitude of the motion, and b is the frequency (number of times the piston moves up and down each second). What effect does doubling the frequency have on the piston's velocity, acceleration, and jerk? (Once you find out, you will know why machinery breaks when you run it too fast.)

68. *Temperatures in Fairbanks, Alaska.* The graph in Fig. 2.42 shows the average Fahrenheit temperature in Fairbanks, Alaska, during a typical 365-day year. The equation that approximates the temperature on day x is

$$y = 37\sin\left[\frac{2\pi}{365}(x - 101)\right] + 25.$$

a) On what day is the temperature increasing the fastest?
b) About how many degrees per day is the temperature increasing when it is increasing at its fastest?

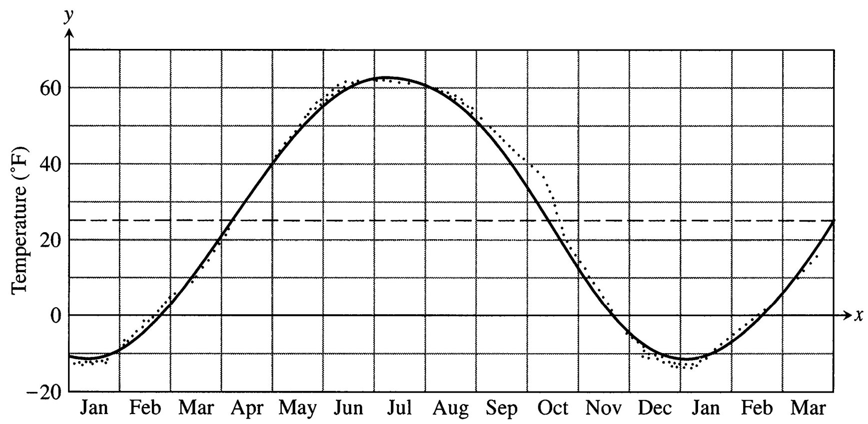
69. The position of a particle moving along a coordinate line is $s = \sqrt{1 + 4t}$, with s in meters and t in seconds. Find the particle's velocity and acceleration at $t = 6$ sec.

70. Suppose the velocity of a falling body is $v = k\sqrt{s}$ m/sec (k a constant) at the instant the body has fallen s meters from its starting point. Show that the body's acceleration is constant.

71. The velocity of a heavy meteorite entering the earth's atmosphere is inversely proportional to $\sqrt{s}$ when it is s kilometers from the earth's center. Show that the meteorite's acceleration is inversely proportional to s^2.

72. A particle moves along the x-axis with velocity $dx/dt = f(x)$. Show that the particle's acceleration is $f(x)f'(x)$.

73. *Temperature and the period of a pendulum.* For oscillations of small amplitude (short swings), we may safely model the relationship between the period T and the length L of a simple



2.42 Normal mean air temperatures at Fairbanks, Alaska, plotted as data points. The approximating sine function is

$$f(x) = 37\sin\left[\frac{2\pi}{365}(x - 101)\right] + 25$$

(Exercise 68).

pendulum with the equation

$$T = 2\pi\sqrt{\frac{L}{g}},$$

where g is the constant acceleration of gravity at the pendulum's location. If we measure g in centimeters per second squared, we measure L in centimeters and T in seconds. If the pendulum is made of metal, its length will vary with temperature, either increasing or decreasing at a rate that is roughly proportional to L. In symbols, with u being temperature and k the proportionality constant

$$\frac{dL}{du} = kL.$$

Assuming this to be the case, show that the rate at which the period changes with respect to temperature is $kT/2$.

74. Suppose that $f(x) = x^2$ and $g(x) = |x|$. Then the composites

$$(f \circ g)(x) = |x|^2 = x^2 \quad \text{and} \quad (g \circ f)(x) = |x^2| = x^2$$

are both differentiable at $x = 0$ even though g itself is not differentiable at $x = 0$. Does this contradict the Chain Rule? Explain.

75. Suppose that $u = g(x)$ is differentiable at $x = 1$ and that $y = f(u)$ is differentiable at $u = g(1)$. If the graph of $y = f(g(x))$ has a horizontal tangent at $x = 1$, can we conclude anything about the tangent to the graph of g at $x = 1$ or the tangent to the graph of f at $u = g(1)$? Give reasons for your answer.

76. Suppose $u = g(x)$ is differentiable at $x = -5$, $y = f(u)$ is differentiable at $u = g(-5)$, and $(f \circ g)'(-5)$ is negative. What, if anything, can be said about the values of $g'(-5)$ and $f'(g(-5))$?

Using the Chain Rule, show that the power rule $(d/dx)x^n = nx^{n-1}$ holds for the functions x^n in Exercises 77 and 78.

77. $x^{1/4} = \sqrt{\sqrt{x}}$

78. $x^{3/4} = \sqrt{x\sqrt{x}}$

Grapher Explorations

79. *The derivative of* $\sin 2x$. Graph the function $y = 2\cos 2x$ for $-2 \le x \le 3.5$. Then, on the same screen, graph

$$y = \frac{\sin 2(x+h) - \sin 2x}{h}$$

for $h = 1.0, 0.5$, and 0.2. Experiment with other values of h, including negative values. What do you see happening as $h \to 0$? Explain this behavior.

80. *The derivative of* $\cos(x^2)$. Graph $y = -2x\sin(x^2)$ for $-2 \le x \le 3$. Then, on the same screen, graph

$$y = \frac{\cos[(x+h)^2] - \cos(x^2)}{h}$$

for $h = 1.0, 0.7$, and 0.3. Experiment with other values of h. What do you see happening as $h \to 0$? Explain this behavior.

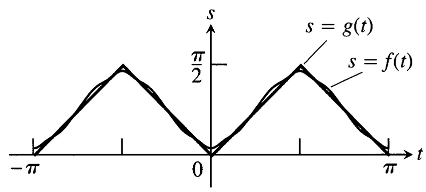
CAS Explorations and Projects

81. As Fig. 2.43 shows, the trigonometric "polynomial"

$$s = f(t) = 0.78540 - 0.63662\cos 2t - 0.07074\cos 6t - 0.02546\cos 10t - 0.01299\cos 14t$$

gives a good approximation of the sawtooth function $s = g(t)$ on the interval $[-\pi, \pi]$. How well does the derivative of f approximate the derivative of g at the points where dg/dt is defined? To find out, carry out the following steps.

a) Graph dg/dt (where defined) over $[-\pi, \pi]$.
b) Find df/dt.
c) Graph df/dt. Where does the approximation of dg/dt by df/dt seem to be best? least good? Approximations by trigonometric polynomials are important in the theories of heat and oscillation, but we must not expect too much of them, as we see in the next exercise.



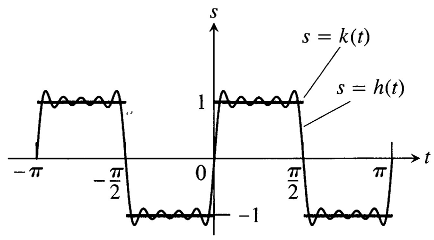
2.43 The approximation of a sawtooth function by a trigonometric "polynomial" (Exercise 81).

82. (*Continuation of Exercise 81.*) In Exercise 81, the trigonometric polynomial $f(t)$ that approximated the sawtooth function $g(t)$ on $[-\pi, \pi]$ had a derivative that approximated the derivative of the sawtooth function. It is possible, however, for a trigonometric polynomial to approximate a function in a reasonable way without its derivative approximating the function's derivative at all well. As a case in point, the "polynomial"

$$s = h(t) = 1.2732\sin 2t + 0.4244\sin 6t + 0.25465\sin 10t + 0.18186\sin 14t + 0.14147\sin 18t$$

graphed in Fig. 2.44 approximates the step function $s = k(t)$ shown there. Yet the derivative of h is nothing like the derivative of k.

a) Graph dk/dt (where defined) over $[-\pi, \pi]$.
b) Find dh/dt.
c) Graph dh/dt to see how badly the graph fits the graph of dk/dt. Comment on what you see.



2.44 The approximation of a step function by a trigonometric "polynomial" (Exercise 82).