Mathematics A1a Lecture Notes

Academic year 2016 autumn

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October 14, 2016

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1 Vectors

Vectors naturally arise when dealing with real-life quantities what cannot be expressed with a single number. Namely: displacement, velocity, force.

Here we mostly deal with 2 dimensional vectors (plane), or 3 dimensional vectors (space), but it is interesting to mention 4 dimensional spaces, such as space-time. In mathematics, abstract vector spaces can have any dimension, even more than finite.

Some of the figures are from Ferenc Wettl's lecture notes.

1.1 Basic definitions

Definition 1.1. A *vector* is a directed line segment, it encodes the relative position between its end and starting points.

A vector is uniquely determined by its *length* and *direction*. Two vectors are the same if and only if they have the same length, they are parallel and have the same orientation. This means that if one translates a vector (both starting and end points), then it stays the same. The null-vector has length 0 and its direction is irrelevant, its staring and end points coinside (starting point = end point).

Let us choose an origin point in the space: O. Then the points in the space can be associated with their position vectors:

$$P \leftrightarrow \overrightarrow{OP}$$

The null vector represents the origin.

Notations:

- A vector: $\vec{v}$ or $\mathbf{v}$
- From starting to end point: $\overrightarrow{AB}$
- length, also called *norm*: $\|\mathbf{v}\|$
- angle of vectors: $\angle(\mathbf{a}, \mathbf{b}) \in [0, \pi]$
- parallel or collinear (same direction, maybe opposite orientation): $\mathbf{a} \parallel \mathbf{b}$
- perpendicular: $\mathbf{a} \perp \mathbf{b}$

Note that the null-vector is parallel and perpendicular to every vector.

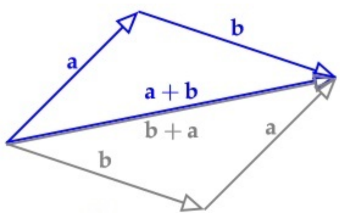
1.2 Operations and their properties

Definition 1.2 (Sum of two vectors)**.** Let $\mathbf{a}$ and $\mathbf{b}$ two vectors, to calculate $\mathbf{a} + \mathbf{b}$ we translate the starting point of $\mathbf{b}$ to the endpoint of $\mathbf{a}$, and the start point of $\mathbf{a} + \mathbf{b}$ is the start point of $\mathbf{a}$ and its end point is the end point of the translated $\mathbf{b}$. In other word: $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.

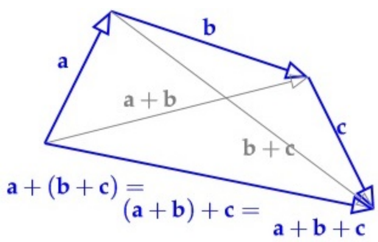
Theorem 1.3 (Commutativity)**.**

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$$

Proof. By parallelogram law. See Figure 1a ☐



(a) commutativity and parallelogram law



(b) Associativity

Figure 1

Theorem 1.4. One can get $\mathbf{a} + \mathbf{b}$ by parallelogram law. In particular, if a parallelogram has two neighbouring sides $\mathbf{a}$ and $\mathbf{b}$, then its diagonal is $\mathbf{a} + \mathbf{b}$.

Proof. By parallelogram law. See Figure 1a. □

Theorem 1.5 (Associativity)**.**

$$\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$$

Proof. See figure 1b □

Definition 1.6 (Scalar multiplication)**.** Let $c \in \mathbb{R}$ a scalar number and $\mathbf{a}$ a vector, we define $c \cdot \mathbf{a}$ by taking the direction of $\mathbf{a}$ and multiply its length by c, reverse the orientation if $c < 0$.

Note that $0 \cdot \mathbf{a} = \mathbf{0}$ for any $\mathbf{a}$. In particular, multiplying with -1 (negation) reverses the orientation: $-\overrightarrow{AB} = \overrightarrow{BA}$. One can subtract vectors as adding its negate. See Figure 2.

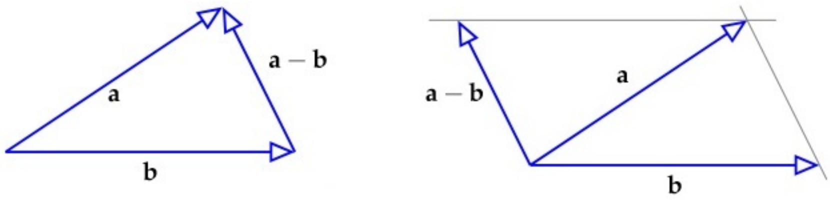


Figure 2: Subtracting vectors

Theorem 1.7 (Properties of the operations)**.** For any vector $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ and scalar numbers r, s the following holds.

commutativity $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$

associativity $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$

additive identity $\mathbf{a} + \mathbf{0} = \mathbf{a}$

additive inverse $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$

compatibility $r \cdot (s \cdot \mathbf{a}) = (r \cdot s) \cdot \mathbf{a}$

distributivity I $r \cdot (\mathbf{a} + \mathbf{b}) = r \cdot \mathbf{a} + r \cdot \mathbf{b}$

distributivity II $(r + s) \cdot \mathbf{a} = r \cdot \mathbf{a} + s \cdot \mathbf{a}$

scalar identity $1 \cdot \mathbf{a} = \mathbf{a}$ and $0 \cdot \mathbf{a} = \mathbf{0}$

□

Definition 1.8 (dot product)**.** Let $\mathbf{a}, \mathbf{b}$ two vectors, then their *dot product* is the following number.

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \cdot \|\mathbf{b}\| \cdot \cos \angle(\mathbf{a}, \mathbf{b})$$

If any of the vectors is a null-vector, then the length is 0 so the angle is irrelevant.

Theorem 1.9. The dot product is symmetric and linear, i.e.

1. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
2. $(c \cdot \mathbf{a}) \cdot \mathbf{b} = c \cdot (\mathbf{a} \cdot \mathbf{b})$
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

Proof. The 1. follows from the definition.
The 2. is trivial.
The 3. See Figure 3. □

Theorem 1.10.

$$\mathbf{a} \perp \mathbf{b} \Leftrightarrow \mathbf{a} \cdot \mathbf{b} = 0$$

Proof. $\cos \angle(\mathbf{a}, \mathbf{b}) = 0$. □

Theorem 1.11.

$$\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$$

or equivalently

$$\sqrt{\mathbf{a} \cdot \mathbf{a}} = \|\mathbf{a}\|$$

Proof. $\cos \angle(\mathbf{a}, \mathbf{a}) = 1$. □

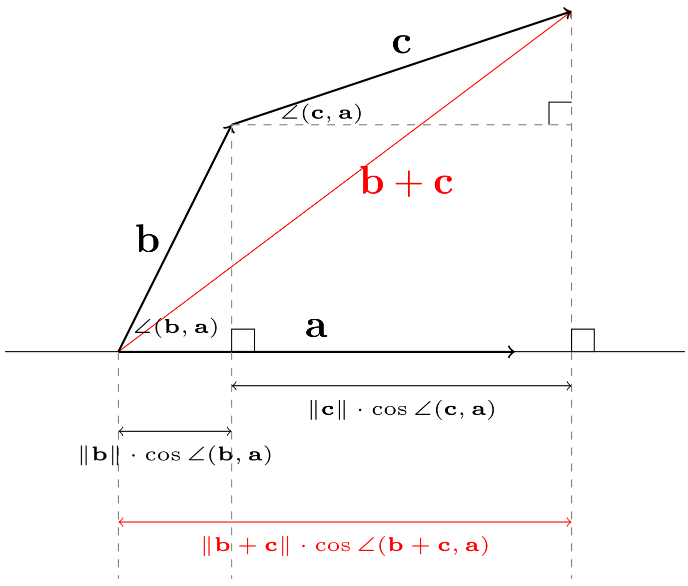


Figure 3: The distributivity of the dot product

Theorem 1.12.

$$\angle(\mathbf{a}, \mathbf{b}) = \arccos \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \cdot \|\mathbf{b}\|}$$

Proof. $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \cdot \|\mathbf{b}\| \cdot \cos \angle(\mathbf{a}, \mathbf{b})$ by definition.

$$\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \cdot \|\mathbf{b}\|} = \cos \angle(\mathbf{a}, \mathbf{b})$$

□

Theorem 1.13 (Projection)**.**

1. If $\|\mathbf{e}\| = 1$ and $\mathbf{b}$ is an other vector, then
$$(\mathbf{e} \cdot \mathbf{b})\mathbf{e}$$
gives the projection of $\mathbf{b}$ to $\mathbf{e}$. Note that the first product is a dot product and the second is a scalar multiplication.

2. Also if $\mathbf{a}$ is any vector (except the null-vector), then
$$\operatorname{proj}_{\mathbf{a}} \mathbf{b} := \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}} \mathbf{a}$$
gives the projection to the direction of $\mathbf{a}$. Note that the fraction is a scalar divided by an other scalar.

3. For the projection $\operatorname{proj}_{\mathbf{a}} \mathbf{b}$ it is true that $(\mathbf{b} - \operatorname{proj}_{\mathbf{a}} \mathbf{b}) \perp \mathbf{a}$.

Proof. For the first part, see Figure 4a.

The second part is as follows. Take $\mathbf{a}' := \frac{\mathbf{a}}{\|\mathbf{a}\|}$. Then $\mathbf{a}'$ has length 1, and we can apply the first part.

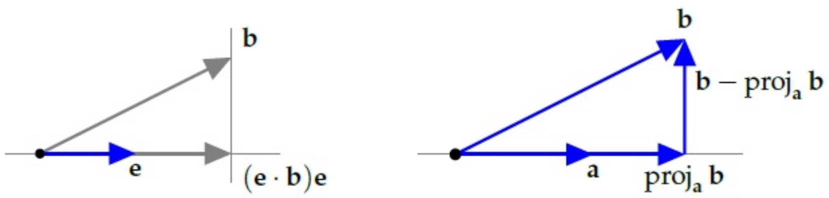
$$(\mathbf{a}' \cdot \mathbf{b})\mathbf{a}' = \left(\frac{\mathbf{a}}{\|\mathbf{a}\|} \cdot \mathbf{b} \right) \frac{\mathbf{a}}{\|\mathbf{a}\|} = \left(\frac{\mathbf{a}}{\sqrt{\mathbf{a} \cdot \mathbf{a}}} \cdot \mathbf{b} \right) \frac{\mathbf{a}}{\sqrt{\mathbf{a} \cdot \mathbf{a}}} = \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}} \mathbf{a}$$

The third part can be explicitly calculated:

$$(\mathbf{b} - \operatorname{proj}_{\mathbf{a}} \mathbf{b}) \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{a} - \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}} \mathbf{a} \right) \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{a} - \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}} \cdot (\mathbf{a} \cdot \mathbf{a}) = \mathbf{b} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} = 0$$

And they are perpendicular as Theorem 1.10 says. □

Definition 1.14 (Right-hand rule)**.** 3 vectors in a 3 dimensional space satisfy the *right-hand rule* if they fit on ones right hand: thumb, index finger and middle finger as the first, second and third vector. See Figure 5



(a) Projection with unit-length vector

(b) Parallel and perpendicular component via projection

Definition 1.15 (cross product)**.** Let $\mathbf{a}, \mathbf{b}$ two vectors in the 3-dimensional space, then their *cross product* is the vector which

1. has length $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \cdot \|\mathbf{b}\| \cdot |\sin \angle(\mathbf{a}, \mathbf{b})|$
2. is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ and
3. its orientation is given by the right-hand rule. See Figure 5

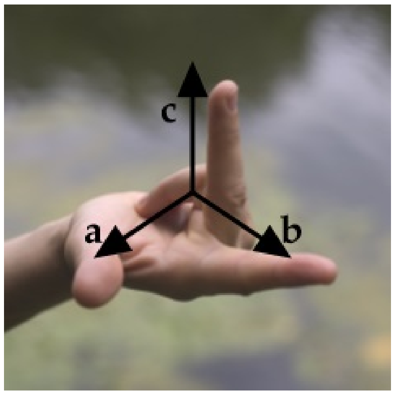


Figure 5: The right-hand rule

Note that it is defined only in 3 dimensions. Also note that $\|\mathbf{a} \times \mathbf{b}\|$ is the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$.

Theorem 1.16. The cross product is anticommutative and linear, i.e.

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. $(c \cdot \mathbf{a}) \times \mathbf{b} = c \cdot (\mathbf{a} \times \mathbf{b})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$

The proof is omitted.

Theorem 1.17. $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

Proof. $\sin \angle(\mathbf{a}, \mathbf{b}) = 0$. □

In particular, $\mathbf{a} \times \mathbf{a} = \mathbf{0}$.

Definition 1.18 (triple product)**.** Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be three vectors in 3 dimensional space. Then the scalar valued *triple product* is the following.

$$(\mathbf{a}\ \mathbf{b}\ \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

Theorem 1.19. $|(\mathbf{a}\ \mathbf{b}\ \mathbf{c})|$ is the volume of the Parallelepiped with edges $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

Proof. □ sketch

Exercise 1.1. What is the volume of the tetrahedron with vertices A, B, C and D?

Theorem 1.20. The triple product is linear in all of its three variables, invariant under circular reordering and anticommutative under non-circular reordering. i.e.

- $((\mathbf{a}_1 + \mathbf{a}_1)\ \mathbf{b}\ \mathbf{c}) = (\mathbf{a}_1\ \mathbf{b}\ \mathbf{c}) + (\mathbf{a}_2\ \mathbf{b}\ \mathbf{c})$, and similarly in every variable.

- $((x \cdot \mathbf{a})\ \mathbf{b}\ \mathbf{c}) = x \cdot (\mathbf{a}\ \mathbf{b}\ \mathbf{c})$, and similarly in every variable.
- $(\mathbf{a}\ \mathbf{b}\ \mathbf{c}) = (\mathbf{b}\ \mathbf{c}\ \mathbf{a}) = (\mathbf{c}\ \mathbf{a}\ \mathbf{b})$
- $(\mathbf{a}\ \mathbf{b}\ \mathbf{c}) = -(\mathbf{c}\ \mathbf{b}\ \mathbf{a})$

Exercise 1.2. Prove the above theorem!

Theorem 1.21. For the triple product $(\mathbf{a}\ \mathbf{b}\ \mathbf{c})$ te following holds.

- $(\mathbf{a}\ \mathbf{b}\ \mathbf{c}) = 0$ if and only if $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are *co-planar*, they are on the same plane.
- $(\mathbf{a}\ \mathbf{b}\ \mathbf{c}) > 0$ if and only if the vectors obey the right-hand rule.
- $(\mathbf{a}\ \mathbf{b}\ \mathbf{c}) < 0$ if and only if the vectors obey the left-hand rule.

1.3 Linear (in)dependence and basis

Definition 1.22 (linear combination)**.** Let $\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_n$ be any vectors and $c_1, c_2, \dots c_n$ any scalar numbers. We call the following a *linear combination.*

$$c_1 \cdot \mathbf{v}_1 + c_2 \cdot \mathbf{v}_2 + \ldots + c_n \cdot \mathbf{v}_n$$

We call the scalars *coefficients.*

Theorem 1.23. Lets take a non null-vector $\mathbf{v}$. Every vector $\mathbf{a}$ which is parallel to $\mathbf{v}$ can be uniquely determined in the form $\mathbf{a} = c \cdot \mathbf{v}$.

Proof. If the vectors $\mathbf{v}$ and $\mathbf{a}$ have the same orientation, then

$$c = \frac{\|a\|}{\|v\|}$$

without dividing by zero.

If they have opposite orientation, then $c = -\frac{\|a\|}{\|v\|}$. □

Also $c \cdot \mathbf{a}$ is parallel with $\mathbf{a}$ for every $c \in \mathbb{R}$.

Theorem 1.24. Lets take two vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ which are not parallel. Then every vector $\mathbf{v}$ in their plane can be uniquely determined in the form

$$\mathbf{v} = v_1 \cdot \mathbf{a}_1 + v_2 \cdot \mathbf{a}_2$$

Proof. To get the scalars v_1 and v_2, use the parallelogram as in Figure 6.

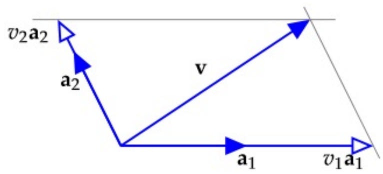


Figure 6: Decomposition of a vector in two dimensions

To get uniqueness, suppose the contrary, namely that

$$\mathbf{v} = v_1 \cdot \mathbf{a}_1 + v_2 \cdot \mathbf{a}_2 = v'_1 \cdot \mathbf{a}_1 + v'_2 \cdot \mathbf{a}_2$$

Then one can get

$$(v_1 - v'_1) \cdot \mathbf{a}_1 = (v_2 - v'_2) \cdot \mathbf{a}_2$$

But $\mathbf{a}_1$ and $\mathbf{a}_2$ are not parallel, so the previous theorem says that the scalar coefficients can be only zero. □

Finding of the coefficients v_1 and v_2 (with respect to the given vectors $\mathbf{a}_1$ and $\mathbf{a}_2$) is called *decomposition.*
Similarly, in 3 dimensions if one takes 3 vectors, which are not in the same plane, then every other vector can be uniquely expressed as the linear combination of the former vectors, see Figure 7.
The former theorems motivates the following definition.

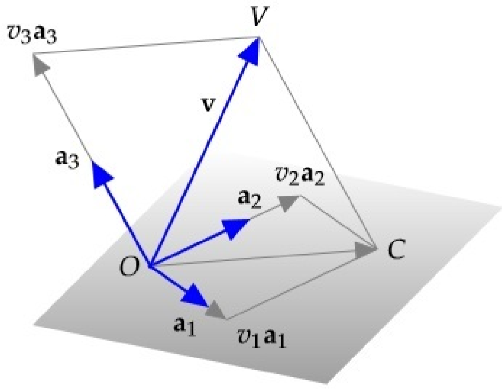


Figure 7: Decomposition of a vector in three dimensions

Definition 1.25 (Linear independence)**.** Let $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n$ be vectors from a vector space (2 or 3 dimensional, or in any dimension). These vectors are called *linearly independent* if no linear combination produces the null-vector, except the trivial one.
In other words:

$$c_1 \cdot \mathbf{v}_1 + c_2 \cdot \mathbf{v}_2 + \ldots + c_n \cdot \mathbf{v}_n = \mathbf{0} \Rightarrow c_1 = c_2 = \ldots = c_n = 0$$

There is an equivalent definition, that no vector cannot be expressed as a linear combination of the others.

Definition 1.26 (Dimension)**.** The *dimension* of a vector space is the maximum number of its independent vectors.

For example on a plane, there is no more than two independent vectors, see Theorem 1.24. Or in a 3 dimensional space, there is no more than 3 independent vectors.

Theorem 1.27 (Unique decomposition)**.** Lets take an independent set of vectors $\mathbf{v_1} \ldots \mathbf{v_n}$ and one which is not independent from the former ones: $\mathbf{x}$. Then $\mathbf{x}$ can be expressed uniquely as the linear combination of the others.

Proof. Being not independent from the others means that $\mathbf{x}$ can be expressed (decomposed) as

$$\mathbf{x} = c_1 \cdot \mathbf{v}_1 + \ldots c_n \cdot \mathbf{v}_n$$

So the only thing to prove is that the coefficients are unique. Let us suppose the contrary:

$$\mathbf{x} = d_1 \cdot \mathbf{v}_1 + \ldots d_n \cdot \mathbf{v}_n$$

then

$$\mathbf{0} = (c_1 - d_1) \cdot \mathbf{v}_1 + \ldots (c_n - d_n) \cdot \mathbf{v}_n$$

and due to independence we get that all of the subtractions have to be 0. ☐

Definition 1.28 (Basis)**.** A maximal set of independent vectors is called a *basis*.

Theorem 1.29 (Coordinates)**.** Given a basis $\{\mathbf{v}_1, \ldots \mathbf{v}_n\}$, one can make a one-to-one correspondace between the vectors and n-tuples of numbers, which are the *coordinates*.

Proof. As mentioned in Theorem 1.27, the coefficients $c_1, \ldots c_n$ are uniquely determined for any vector. Also, given the coefficients, the linear combination is defined uniquely. ☐

The n-tuple $(c_1, \ldots c_n)$ is called the coordinates of the vector (with respect to the basis $\{\mathbf{v}_1, \ldots \mathbf{v}_n\}$). Note that the coordinates are relative up to a given basis.

Exercise 1.3. Szép Gabriella's example: determine the ratio $AM : MF$ on Figure 8!

1.4 Standard basis

Let $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$ be the unit length coordinate vectors in the 3 dimensional space. Sometimes called $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ respectively. Note that $\|\mathbf{i}\| = \|\mathbf{j}\| = \|\mathbf{k}\| = 1$, any two of them are perpendicular and they are ordered according to right-hand rule.

Note that the Theorem 1.29 yield for the standard basis that $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$. Moreover, one can define $\mathbb{R}^n$ only with coordinates, without abstract vectors.

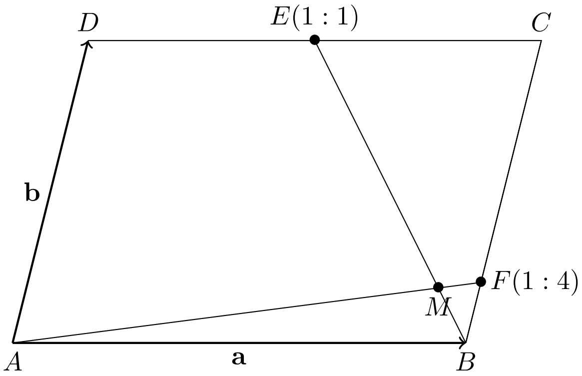


Figure 8

Definition 1.30 (Determinant)**.** For 2 and 3 dimensions see https://en.wikipedia.org/wiki/Determinant.

Theorem 1.31 (Operations in the standard coordinates)**.** Let $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$ and $\mathbf{c} = (c_1, c_2, c_3)$ be three vectors and their standard coordinates and $x \in \mathbb{R}$ a scalar. The operations can be calculated as follows:

- $x \cdot \mathbf{a} = (x \cdot a_1, x \cdot a_2, x \cdot a_3)$
- $\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$
- $\mathbf{a} \cdot \mathbf{b} = a_1 \cdot b_1 + a_2 \cdot b_2 + a_3 \cdot b_3$
- $\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2}$
- $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$
- $(\mathbf{a}\ \mathbf{b}\ \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Proof.

- $$x \cdot \mathbf{a} = x \cdot (a_1 \cdot \mathbf{i} + a_2 \cdot \mathbf{j} + a_3 \cdot \mathbf{k}) = x \cdot a_1 \cdot \mathbf{i} + x \cdot a_2 \cdot \mathbf{j} + x \cdot a_3 \cdot \mathbf{k} = (x \cdot a_1, x \cdot a_2, x \cdot a_3)$$
- $$\mathbf{a}+\mathbf{b} = (a_1 \cdot \mathbf{i} + a_2 \cdot \mathbf{j} + a_3 \cdot \mathbf{k}) + (b_1 \cdot \mathbf{i} + b_2 \cdot \mathbf{j} + b_3 \cdot \mathbf{k}) = ((a_1+b_1) \cdot \mathbf{i} + (a_2+b_2) \cdot \mathbf{j} + (a_3+b_3) \cdot \mathbf{k}) = (a_1+b_1, a_2+b_2, a_3+b_3)$$
- $$\begin{aligned} \mathbf{a} \cdot \mathbf{b} = (a_1 \cdot \mathbf{i} + a_2 \cdot \mathbf{j} + a_3 \cdot \mathbf{k}) \cdot (b_1 \cdot \mathbf{i} + b_2 \cdot \mathbf{j} + b_3 \cdot \mathbf{k}) = \\ a_1 \cdot b_1 \cdot \mathbf{i} \cdot \mathbf{i} + a_1 \cdot b_2 \cdot \mathbf{i} \cdot \mathbf{j} + a_1 \cdot b_3 \cdot \mathbf{i} \cdot \mathbf{k} + \\ a_2 \cdot b_1 \cdot \mathbf{j} \cdot \mathbf{i} + a_2 \cdot b_2 \cdot \mathbf{j} \cdot \mathbf{j} + a_2 \cdot b_3 \cdot \mathbf{j} \cdot \mathbf{k} + \\ a_3 \cdot b_1 \cdot \mathbf{k} \cdot \mathbf{i} + a_3 \cdot b_2 \cdot \mathbf{k} \cdot \mathbf{j} + a_3 \cdot b_3 \cdot \mathbf{k} \cdot \mathbf{k} = a_1 \cdot b_1 + a_2 \cdot b_2 + a_3 \cdot b_3 \end{aligned}$$

□ the rest

Theorem 1.32. Let $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$ and $\mathbf{c} = (c_1, c_2, c_3)$ be three vectors in the space. They are independent, in fact they form a basis, iff

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \neq 0$$

1.5 Homeworks due to Sept. 15.

Exercise 1.4. Let A, B, C and D the vertices of the bottom side of a cube, and A_1, B_1, C_1 and D_1 the vertices of the upper side of the cube. Determine the vector $\overrightarrow{AB} + \overrightarrow{AC_1} + \overrightarrow{BD_1} + \overrightarrow{C_1B}$ with starting and end points.

Exercise 1.5. Given three points $A(1, 0, 1)$, $B(0, 2, 1)$ and $C(0, 1, 2)$

1. Find the area of the triangle ABC!
2. Find the angle at the vertex A!

Exercise 1.6. Let $\|\mathbf{a}\| = 2$, $\|\mathbf{b}\| = 3$ and $\angle(\mathbf{a}, \mathbf{b}) = \frac{2\pi}{3}$. Calculate

$$\|(\mathbf{a} + \mathbf{b}) \times (2\mathbf{a} - \mathbf{b})\| = ?$$

1.6 Homeworks due to Sept. 22.

Exercise 1.7. Let $\mathbf{u} = (1, 2, 1)$, $\mathbf{v} = (0, 1, -1)$ and $\mathbf{w} = (1, 0, 0)$. Calculate

$$(\mathbf{w} \times \mathbf{v}) \times \mathbf{u}$$

Exercise 1.8.

1. What is the angle of the vectors $(2, -1)$ and $(-1, 3)$?
2. What is the right value of t where the angle of the vectors $(1, t, 1)$ and $(t, -1, 1)$ is exactly $60°$?

2 Analytic geometry

In this section we make use of the coordinates and we calculate all sorts of geometric problems with them. These calculations can be done soloely with coordinates (numbers), without any actual geometry, this is why the name *analytical*.

2.1 Equations of line and plane

Definition 2.1. A line can be defined by a point P on the line and a (non-zero) direction vector $\mathbf{v}$. The *parametric vector equation* of the line is

$$P + t\mathbf{v} \quad t \in \mathbb{R}$$

The *parametric coordinate equation* of the line is

$$\begin{pmatrix} P_1 + t \cdot v_1 \\ P_2 + t \cdot v_2 \\ P_3 + t \cdot v_3 \end{pmatrix} \quad t \in \mathbb{R}$$

This means that for every number $t \in \mathbb{R}$ one can get a point on the line.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} P_1 + t \cdot v_1 \\ P_2 + t \cdot v_2 \\ P_3 + t \cdot v_3 \end{pmatrix}$$

If $v_1 \neq 0$, $v_2 \neq 0$ and $v_3 \neq 0$ then

$$t = \frac{x - P_1}{v_1}$$
$$t = \frac{y - P_2}{v_2}$$
$$t = \frac{z - P_3}{v_3}$$

Definition 2.2. The *system of equations* of the line is

$$\frac{x-P_1}{v_1} = \frac{y-P_1}{v_2} = \frac{z-P_3}{v_3}$$

If $v_1 = 0$, then $x = P_1$ and we exclude the first equation. If $v_2 = 0$, then

$$\frac{x-P_1}{v_1} = \frac{z-P_3}{v_3} \text{ and } y = P_2$$

remains.

These formulas also work in 2 dimensions, except that there is no z and third coordinate.

Definition 2.3. A plane can be determined with a point P and two independent vectors on it: $\mathbf{v}_1$ and $\mathbf{v}_2$. The *parametric vector equation* of the plane is

$$P + t\mathbf{v}_1 + s\mathbf{v}_2 \quad t, s \in \mathbb{R}$$

For every value of t and s one gets a point on the plane.

Definition 2.4. A plane can be determined with a point P and one normal vector $\mathbf{n}$, a vector which is perpendicular to the plane. The *vector equation* of the plane is

$$(\mathbf{x} - P) \cdot \mathbf{n} = 0$$

With coordinates, the *coordinate equation* of the plane is

$$(x - P_1) \cdot n_1 + (y - P_2) \cdot n_2 + (z - P_3) \cdot n_3 = 0$$

This means that every point (x, y, z), which fulfills this equation, is on the plane.

Given two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, one can get a normal vector by the cross product.

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2$$

Also, $-\mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{v}_2 \times \mathbf{v}_1$ is good as well.

Theorem 2.5 (Distance between point and line)**.** Let e be a line through point P with direction $\mathbf{v}$ and an other point Q anywhere. Then the distance between Q and e is the following.

$$\text{distance}(Q, e) = \|(Q - P) - \text{proj}_{\mathbf{v}}(Q - P)\|$$

Proof. At first, let us suppose that $P = \mathbf{0}$. Then we simply project the pont Q to the line, and then we calculate the distance between the original and the projected point. See Figure 9.

$$\| \underbrace{Q - \underbrace{\text{proj}_{\mathbf{v}} Q}_{\text{projected point}}}_{\text{original}-\text{projected}} \|$$

For the second part, if P is not zero, then we translate both Q and the line e with $-P$ and then we calculate

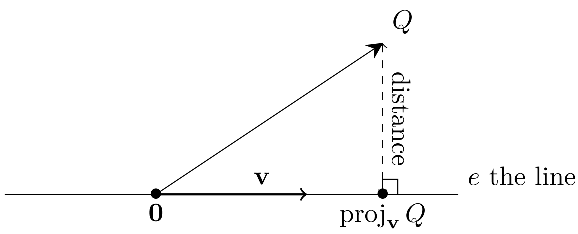


Figure 9: Distance from a line ($P = \mathbf{0}$)

like in the first part. The translated point is $Q - P$ and the translated line has the same direction, but its basepoint is $P - P = \mathbf{0}$. This means that we apply the above formula for $Q - P$:

$$\|(Q - P) - \text{proj}_{\mathbf{v}}(Q - P)\|$$

□

Theorem 2.6 (Distance between point and plane)**.** Let S be a plane with basepoint P and normal vector $\mathbf{n}$ and an other point Q anywhere. Then the distance between Q and S is the following.

$$\text{distance}(Q, S) = \frac{|(Q - P) \cdot \mathbf{n}|}{\|\mathbf{n}\|}$$

Note that there is a dot product and an absolute value in the enumerator.

Proof. See Figure 10.

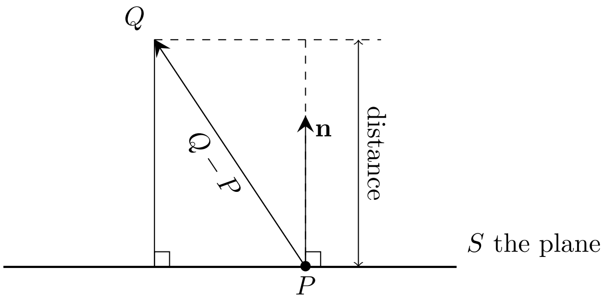


Figure 10: Distance from a plane

The length of the projected vector gives the distance.

$$\|\operatorname{proj}_{\mathbf{n}}(Q - P)\| = \left\| \frac{(Q - P) \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}} \mathbf{n} \right\| = \left| \frac{(Q - P) \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}} \right| \cdot \|\mathbf{n}\| = \left| \frac{(Q - P) \cdot \mathbf{n}}{\|\mathbf{n}\|^2} \right| \cdot \|\mathbf{n}\| = \frac{|(Q - P) \cdot \mathbf{n}|}{\|\mathbf{n}\|}$$

□

Example 2.1. Take three points in the space $A(1, 2, 3), B(2, 3, 4)$ and $C(0, 1, 0)$. What is the plane through these points?

Solution. For determining the plane, we give a basepoint and a normal vector. Lets choose A as a basepoint (any of the three points are good). For the normal vector

$$\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = (B - A) \times (C - A) = \left(\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right) \times \left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right) =$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ -1 & -1 & -3 \end{vmatrix} = \mathbf{i} \cdot \begin{vmatrix} 1 & 1 \\ -1 & -3 \end{vmatrix} - \mathbf{j} \cdot \begin{vmatrix} 1 & 1 \\ -1 & -3 \end{vmatrix} + \mathbf{k} \cdot \begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} = \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix}$$

From these we can write

$$(\mathbf{x} - A) \cdot \mathbf{n} = 0$$

which is

$$(x - 1) \cdot (-2) + (y - 2) \cdot 2 + (z - 3) \cdot 0 = 0$$

or the same as

$$-2x + 2y = 2 \text{ or } -x + y = 1$$

□

2.2 Homeworks due to Sept. 29.

Choose 3 out of 6!

Exercise 2.1. Determine the line with parametric equation and system of equations!

- The line which goes through the point $A(-2, 5, 1)$ and has a direction $\mathbf{v}(-1, 2, 3)$.
- The line which is parallel to $\mathbf{j}(0, 1, 0)$ and goes through point $A(5, 1, 4)$.
- The line which goes through points $P(3, 1, 2)$ and $Q(-1, 1, 3)$.

Exercise 2.2. What is the basepoint and normal vector of the plane with the following equation?

$$x + 2y + 2z = 13$$

What is the distance of the origin (the point $(0,0,0)$) from this plane?

Exercise 2.3.

- Determine the plane (with its equation) which goes through the origin and is perpendicular to the line

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+3}{4}$$

- Determine the line (with both parametric equation and system of equations) which goes through the point $A(0,7,0)$ and is perpendicular to the plane

$$7x - y + 3z = 0$$

Exercise 2.4. Determine the plane, which goes through the points $A(-1,2,-3), B(6,-2,-3)$ and is parallel to the line $\frac{x-1}{3} = \frac{y+1}{2} = \frac{3-z}{5}$!

Exercise 2.5.

- Are these points on the same line?

$$A(-2,3,1), B(0,5,2), C(-4,1,0)$$

- Are these points on the same plane?

$$A(-3,0,4), B(4,1,2), C(0,0,0), D(5,2,1)$$

Explain why!

Exercise 2.6. What are the coordinates of the point M on Figure 11?

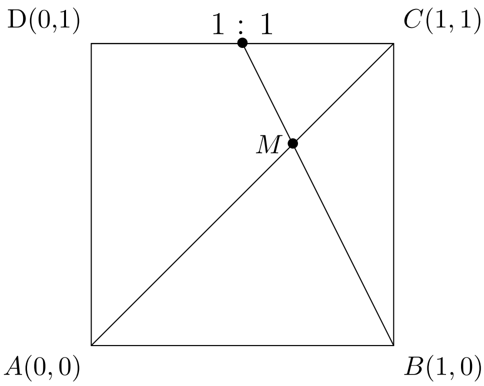


Figure 11

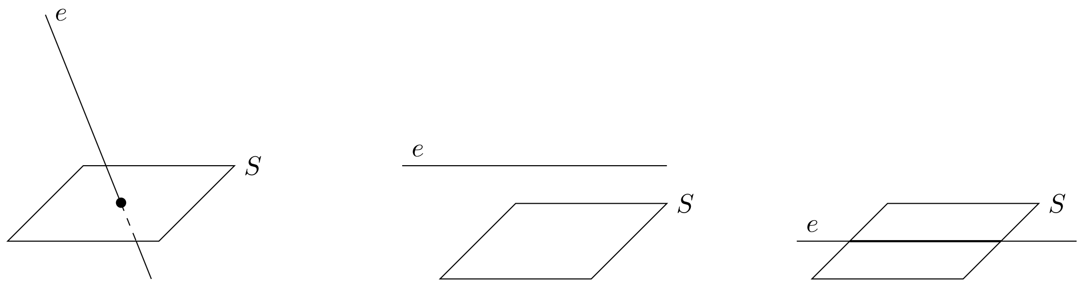
2.3 Relative positions

Theorem 2.7 (Line and plane)**.** Let S be a plane and e be a line in the space. Let $\mathbf{n}$ be the normal vector of the plane and $\mathbf{v}$ be the direction of the line. Their *relative position* can be one of the followings.

intersecting They have exactly one common point. Figure 12a.

parallel They have no common point. Figure 12b.

coinside The line is on the plane (they have infinitely many common points). Figure 12c.



(a) intersecting (b) parallel (c) coinside

Figure 12: Possible positions of a line and a plane

The parallel and the coinside cases can occur only if the direction of e is perpendicular to the normal vector of S. With formulas:

$$\mathbf{v} \cdot \mathbf{n} = 0$$

To check whether they have a common point or not, one should try to solve the equation of the plane and the system of equations of the line together.

Theorem 2.8 (Line and line)**.** Let e and f be two lines in the space with direction vectors $\mathbf{v}_e$ and $\mathbf{v}_f$. Their *relative position* can be one of the followings.

intersecting They have exactly one common point. Figure 13a.

parallel They have no common point, but they are parallel. Figure 13b.

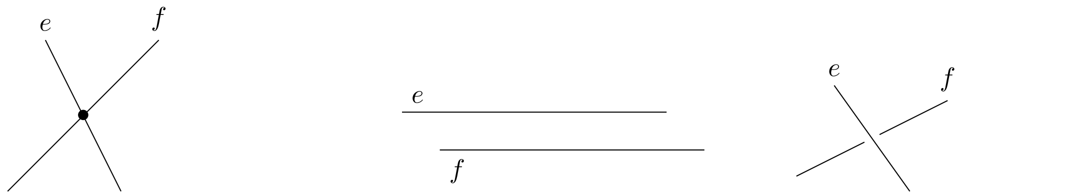
skew They have no common point, and they are not parallel either. Figure 13c.

coinside The two lines are actually the same.

The parallel and the coinside cases can occur only if the direction of e is parallel to the direction of f. With formula:

$$\mathbf{v}_e = a \cdot \mathbf{v}_f \quad \text{for some } a \in \mathbb{R}$$

To check whether they have a common point or not, one should try to solve the two system of equations together.



(a) intersecting (b) parallel (c) skew

Figure 13: Possible positions of a line and a plane

Theorem 2.9 (Plane and plane)**.** Two planes in the space can have the following relative positions.

intersecting They have some common points.

parallel They have no common point, but they are parallel.

coinside They are actually the same.

The parallel and the coinside cases can occur only if the normal vectors are parallel. If they intersect then the common part is a line. To get the intersection, one have to solve the two equations of the lines as a system of equations.

Example 2.2. Let $S : 2x + y - z = 3$ and $T : x + 2y + z = 2$ be two planes. What is their relative position? What is their intersection?

Solution. The normal vector of the first one is $\mathbf{n}_S(2,1,-1)$ and for the second one $\mathbf{n}_T(1,2,1)$. Those two vectors are clearly independent, so they are not on the same line. (you can check it by $\mathbf{n}_S \times \mathbf{n}_T \neq \mathbf{0}$) This means that the planes must intersect and the intersection must be a line. To determine the intersection one should solve the following.

$$2x + y - z = 3$$
$$x + 2y + z = 2$$

If I add the two equations, then I get $3x + 3y = 5$ or $x + y = \frac{5}{3}$. If I subtract them, then I get $x - y - 2z = 1$ or $x - y = 1 + 2z$.

$$x + y = \frac{5}{3}$$
$$x - y = 1 + 2z$$

If I add the latter two, then I get $2x = \frac{5}{3} + 1 + 2z$ or $x = \frac{4}{3} + z$. If I subtract them, then $2y = \frac{5}{3} - 1 - 2z$ or $y = \frac{1}{3} - z$. So what I got is the following.

$$x = \frac{4}{3} + z$$
$$y = \frac{1}{3} - z$$
$$z \in \mathbb{R}$$

This gives the parametric equation of the intersection:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4/3 \\ 1/3 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

□

2.4 Homeworks due to Oct. 06.

You have to do at least 3 out of 6!

Exercise 2.7. Determine the distance between the following objects.

- The point $(0,0,12)$ and the line $x = 4t, y = -t, z = 2t$.
- The point $(0,-1,0)$ and the plane $2x + y + 2z = 4$.

Exercise 2.8. Determine the intersection point of the following objects, if there is any.

- The line $x = 3 - t, y = 2 - t, z = 3 - t$ and the plane $-2x + y + 3z - 3 = 0$.
- The line $x + 2 = y - 3 = \frac{z+1}{3}$ and the plane $x + 2y - z + 2 = 0$.

Exercise 2.9. Does the following lines intersect? If so, what is the angle between them? Also calculate the plane through them!

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \cdot \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \text{ and } \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + t \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Exercise 2.10. What is the angle between the following planes?

$$S : 2x + y - z = 3 \text{ and } T : x + 2y + z = 2$$

Exercise 2.11. Determine the planes through the sides of the Tetrahedron with vertices

$$A(0,0,0), B(1,0,0), C(0,1,0), D(0,0,1)$$

The anwer is four equations, because the Tetrahedron has four sides!

Exercise 2.12. Determine the equidistant plane of the points $A(1,0,2)$ and $B(0,1,1)$!

3 Sequences

The following section deals with infinite sequences of numbers. In particular, we are interested in the behaviour of these numbers at infinity. The study of these infinite kind is the first step in calculus and analysis.

Some parts of the material are from Attila Andai's lecture notes.

3.1 Basic concepts

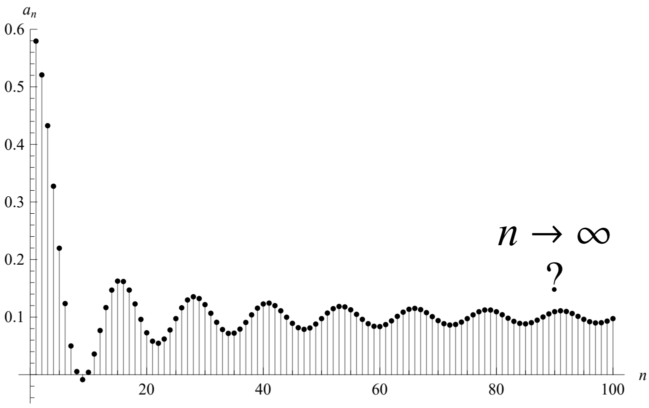


Figure 14: We are interested in the behaviour of these numbers at infinity

Definition 3.1 (Sequence)**.** A *sequence* is a $\mathbb{N} \mapsto \mathbb{R}$ function. For the function $a : \mathbb{N} \mapsto \mathbb{R}$ we usually denote $a(n)$ (spelled as "a of n") with a_n, the n^{th} element in the sequence.

Definition 3.2 (Limit)**.** We say that the sequence $a : \mathbb{N} \mapsto \mathbb{R}$ has the *limit* $A \in \mathbb{R}$ if

for any $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that whenever $n > N$ then $|a_n - A| < \varepsilon$ also holds.

With formulas:

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \text{ such that } \forall n (N < n \Rightarrow |a_n - A| < \varepsilon)$$

The sequence's limit is infinity (∞) if it grows beyond any finite number:

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \text{ such that } \forall n (N < n \Rightarrow a_n > \varepsilon)$$

The sequence's limit is negative infinity ($-\infty$) if it decreases below any number:

$$\forall \varepsilon < 0 \; \exists N \in \mathbb{N} \text{ such that } \forall n (N < n \Rightarrow a_n < \varepsilon)$$

If a sequence has a finite limit, then we say that it is *convergent*, otherwise *divergent*.

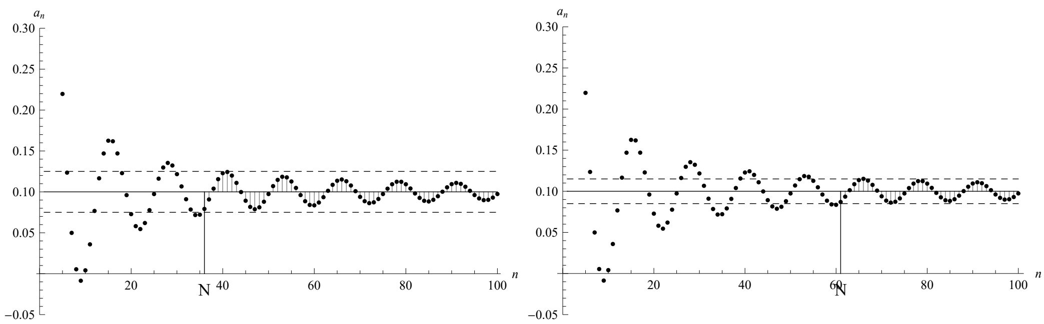


Figure 15: Limit

Example 3.1. Let $a_n = \frac{1}{n}$ then prove that $\lim_{n \to \infty} a_n = 0$.

Proof. To check whether the definition 3.2 holds, one has to determine N for every $\varepsilon > 0$ number. Let us suppose that $\varepsilon > 0$ is given and we want to determine when

$$|a_n - A| < \varepsilon$$

does hold. In this example:

$$\begin{aligned} \left|\frac{1}{n} - 0\right| &< \varepsilon \\ &\Updownarrow \\ \frac{1}{n} &< \varepsilon && / \cdot \frac{n}{\varepsilon} \\ &\Updownarrow \\ \frac{1}{\varepsilon} &< n \end{aligned}$$

This concludes that for $N = \frac{1}{\varepsilon}$ the definition is satisfied. Actually N may not be an integer, so we should write its rounded value instead: $N = \text{floor}(\frac{1}{\varepsilon})$. □

Example 3.2. Let $a_n = \frac{1}{n^2}$ then prove that $\lim_{n\to\infty} a_n = 0$.

Proof. Like before, one has to solve

$$|a_n - A| < \varepsilon$$

for n with a given ε. In this example:

$$\begin{aligned} \left|\frac{1}{n^2} - 0\right| &< \varepsilon \\ &\Updownarrow \\ \frac{1}{n^2} &< \varepsilon && / \cdot \frac{n^2}{\varepsilon} \\ &\Updownarrow \\ \frac{1}{\varepsilon} &< n^2 && /\sqrt{\ } \\ &\Updownarrow \\ \sqrt{\frac{1}{\varepsilon}} &< n \end{aligned}$$

This concludes that for $N = \sqrt{\frac{1}{\varepsilon}}$ the definition is satisfied. □

Theorem 3.3 (Constant sequence)**.** Let $a_n = a \in \mathbb{R}$ a constant sequence. Then $\lim_{n\to\infty} a_n = a$.

Proof. For any $\varepsilon > 0$ the index $N = 0$ is sufficient.

$$|a_n - a| = |a - a| = 0 < \varepsilon \text{ no matter what } n \text{ is}$$

□

Theorem 3.4.

$$\lim_{n\to\infty} a_n = 0 \Leftrightarrow \lim_{n\to\infty} |a_n| = 0$$

Proof. The convergence is equivalent to

$$|a_n - 0| < \varepsilon \quad \text{from a given number } n > N$$

In which

$$|a_n - 0| = |a_n| = \big||a_n| - 0\big|$$

Formally this means that $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} |a_n| = 0$ should be both true, or both false. In other words they are equivalent. □

In the following theorems I will use the so called triangle inequality, see the Theorem 4.1 in the Appendix.

Theorem 3.5 (Uniqueness of the limit)**.** If $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} a_n = B$ then $A = B$.

This means that a convergent sequence has exactly one limit.

Proof. Let us suppose the contrary, namely that $A \neq B$. We will find that this is impossible. If $A \neq B$ then $\frac{|A-B|}{2} > 0$, let $\varepsilon := \frac{|A-B|}{2}$. By the definition of the limit, there should be an $N \in \mathbb{N}$ such that for every $n > N$

$$|a_n - A| < \varepsilon \text{ and } |a_n - B| < \varepsilon$$

hold. If so, then the following reasoning should be true.

$$\frac{|A-B|}{2} = \frac{|(a_n - B) - (a_n - A)|}{2} \leq \frac{|(a_n - B)| + |(a_n - A)|}{2} < \frac{\frac{|A-B|}{2} + \frac{|A-B|}{2}}{2} = \frac{|A-B|}{2}$$

This chain of inequalities would mean that $\frac{|A-B|}{2} < \frac{|A-B|}{2}$, which is wrong ↯. This means that my original assumption should be false: $A \neq B$ is false, therefore $A = B$. □

Definition 3.6. Let a_n be a sequence.

a_n is *bounded from above* if there exists a $K \in \mathbb{R}$ such that for every $n \in \mathbb{N}$

$$a_n \leq K.$$

In this case K is an *upper bound*.

The sequence is *bounded from below* if there exists a $K \in \mathbb{R}$ such that for every $n \in \mathbb{N}$

$$a_n \geq K.$$

In this case K is an *lower bound*.

The sequence is *bounded* if it is bounded from both above and below.

Theorem 3.7. If $\lim_{n\to\infty} a_n = A$ then a_n is bounded.

Proof. Let $\varepsilon = 1$, for this ε one should be able to find an index $N \in \mathbb{N}$ such that

$$|a_n - A| < \varepsilon = 1$$

for all $n > N$. This means that

$$A - 1 \leq a_n \leq A + 1 \quad \text{for } n > N.$$

This means that a_n is bounded for indices $n > N$. What about $0 \leq n \leq N$?

$$K_1 := \max_{0 \leq n \leq N} a_n$$
$$K_2 := \min_{0 \leq n \leq N} a_n$$

For these numbers it is true that

$$a_n \leq \max(K_1, A + 1) \text{ and}$$
$$a_n \geq \min(K_2, A - 1)$$

So there is an upper and a lower bound. □

Note that if there is an upper and a lower bound, then one can come up with a common bound. If

$$a_n \leq K_1 \text{ and}$$
$$a_n \geq K_2$$

then $K := \max(|K_1|, |K_2|)$ and then

$$-K \leq a_n \leq K$$
$$\Updownarrow$$
$$|a_n| \leq K$$

Theorem 3.8. Let b_n be a bounded sequence and we have a sequence $\lim_{n\to\infty} a_n = 0$. Then

$$\lim_{n\to\infty} a_n \cdot b_n = 0$$

Proof. b_n is bounded, this means that there exists a $K > 0$ such that

$$|b_n| \leq K \quad \forall n \in \mathbb{N}$$

Let $\varepsilon > 0$ be any positive number, the convergence of a_n says that there is an $N \in \mathbb{N}$ such that $|a_n| < \frac{\varepsilon}{K}$ for $n > N$. Then we have that

$$|a_n \cdot b_n| = |a_n| \cdot |b_n| \leq |a_n| \cdot K < \frac{\varepsilon}{K} \cdot K = \varepsilon \quad \text{if } n > N$$

□

Theorem 3.9. If $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$ then

$$\lim_{n\to\infty} a_n + b_n = A + B$$

We say that *the limit of the sum is the sum of the limits* or *the addition is continuous*!

Proof. If both a_n and b_n are convergent, then for every $\varepsilon > 0$ I can find an index N such that for every $n > N$ the followings hold.

$$|a_n - A| < \frac{\varepsilon}{2} \text{ and } |b_n - B| < \frac{\varepsilon}{2}$$

With that I can conclude that

$$|a_n + b_n - (A + B)| = |(a_n - A) + (b_n - B)| \leq |a_n - A| + |b_n - B| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

In this way I could find the index N for any given ε for the sequence $a_n + b_n$. □

Theorem 3.10. If $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$ then

$$\lim_{n\to\infty} a_n \cdot b_n = A \cdot B$$

We say that *the limit of the product is the product of the limits* or *the multiplication is continuous*!

Proof. We assumed that both a_n and b_n are convergent. Theorem 3.7 says that a_n is bounded:

$$\exists K > 0 \text{ such that } |a_n| \leq K.$$

Also for every $\varepsilon > 0$ I can find an index N such that for every $n > N$ the followings hold.

$$|a_n - A| < \frac{\varepsilon}{K + |B|} \text{ and } |b_n - B| < \frac{\varepsilon}{K + |B|}$$

With these I can conclude that

$$|a_n \cdot b_n - A \cdot B| = |a_n \cdot b_n - a_n \cdot B + a_n \cdot B - A \cdot B| = |a_n \cdot (b_n - B) + (a_n - A) \cdot B| \leq$$

$$|a_n \cdot (b_n - B)| + |(a_n - A) \cdot B| = |a_n| \cdot |b_n - B| + |a_n - A| \cdot |B| \leq$$

$$K \cdot |b_n - B| + |a_n - A| \cdot |B| < K \cdot \frac{\varepsilon}{K + |B|} + |B| \cdot \frac{\varepsilon}{K + |B|} = \varepsilon$$

In this way I could find the index N for any given ε for the sequence $a_n \cdot b_n$. □

Theorem 3.11. If $\lim_{n\to\infty} a_n$ is convergent and $c \in \mathbb{R}$ is a number, then

$$\lim_{n\to\infty} c \cdot a_n = c \cdot \lim_{n\to\infty} a_n$$

□

Theorem 3.12. If a_n is convergent and the limit is not zero, then

$$\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{\lim_{n\to\infty} a_n}$$

We can say that the *division is continuous*, as long as the denominator is not zero. □

This also concludes that one can divide two convergent sequences and their limits as long as the denominator is not zero.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$$

Theorem 3.13. If $\lim_{n\to\infty} a_n = A$ and $p \in \mathbb{N}$ then

$$\lim_{n\to\infty} (a_n)^p = A^p$$

We can say that the p^{th} *power is continuous.* □

Theorem 3.14. If $\lim_{n\to\infty} a_n = A$ and $q \in \mathbb{N}, q \geq 2$ and $A > 0$ then

$$\lim_{n\to\infty} \sqrt[q]{a_n} = \sqrt[q]{A}$$

We can say that the q^{th} *root is continuous.* □

Theorem 3.15. If the sequence a_n is monotone increasing and bounded from above, then it is convergent.

Proof. □

Theorem 3.16 (Monotonity)**.** Let a_n and b_n are two convergent sequences, then

$$a_n \leq b_n \ \forall n \in \mathbb{N} \ \Rightarrow \ \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$$

Proof. Let us suppose the contrary, namely that $A > B$. In this case I will get a contradiction. If $A > B$ then $\varepsilon := \frac{A-B}{2}$ and choose $N \in \mathbb{N}$ such that

$$|a_n - A| < \varepsilon \text{ and } |b_n - B| < \varepsilon \quad \text{for } n > N$$

Then

$$A - \varepsilon < a_n < A + \varepsilon$$
$$B - \varepsilon < b_n < B + \varepsilon$$

And

$$A - \varepsilon = A - \frac{A-B}{2} = \frac{A+B}{2} < a_n$$
$$b_n < B + \varepsilon = B + \frac{A-B}{2} = \frac{A+B}{2}$$

Which means that

$$b_n < \frac{A+B}{2} < a_n$$

whish is a contradistion since we assumed that $a_n \leq b_n$ ↯.

This means that my original assumption that $A > B$ should be false, therefore

$$A \leq B$$

□

Theorem 3.17 (Squeeze theorem)**.** If there are three sequences a_n, b_n and c_n such that.

$$a_n \leq b_n \leq c_n$$

for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = x \ in \mathbb{R}$$

then b_n is also convergent and

$$\lim_{n\to\infty} b_n = x$$

Proof. Let $b'_n := b_n - a_n$ and $c'_n := c_n - a_n$ then

$$0 \leq b'_n \leq c'_n$$

Since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n$ therefore $\lim_{n\to\infty} c'_n = 0$. For any $\varepsilon > 0$ there exists an index $N \in \mathbb{N}$ such that for $n > N$

$$c'_n < \varepsilon$$

In this case

$$0 \leq b'_n \leq c'_n < \varepsilon$$

So I concluded that N is a good index for ε, which concludes that $\lim_{n\to\infty} b'_n = 0$. From that I can say that

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} (b'_n + a_n) = \lim_{n\to\infty} b'_n + \lim_{n\to\infty} a_n = 0 + x = x$$

□

Theorem 3.18. If $a_n \leq b_n$ are two sequences and $\lim_{n\to\infty} a_n = \infty$ then

$$\lim_{n\to\infty} b_n = \infty$$

□

3.2 Homeworks due to Oct. 13.

You have to do at least 3 out of 6! Give an explanation (proof) how you came up with the answer, not just the answer!

Example 3.3. Let $a_n = 1 + \frac{1}{n}$ what is the limit?

Example 3.4. Let $a_n = \frac{1}{n} + n$, is it bounded from above or from below or neither or both?

Example 3.5. Let $a_n = \frac{(-1)^n}{n}$, is it convergent? If so, what is the limit?

Example 3.6. Let $a_n = \frac{\sin(n)}{n}$, is it convergent? If so, what is the limit?

Example 3.7. Let $a_n = n^2 - n$, is it bounded from above or from below or neither or both?

Example 3.8. Let a_n be defined in the following way.

$$\begin{aligned} a_0 &= 0 \\ a_1 &= 0.9 \\ a_2 &= 0.99 \\ a_3 &= 0.999 \\ &\vdots \\ a_n &= 0.\underbrace{999\ldots9}_{n \text{ times}} \end{aligned}$$

Is it convergent, if so what is the limit?

3.3 Exercises for Oct. 20.

You don't have to do any of them, it is for practicing!

Example 3.9. Calculate the limit of the followings

1. $a_n = \frac{n}{n^2+1}$
2. $a_n = \frac{n+1}{n-1}$

You can find some exaples with solutions here:

Limits http://www.vitutor.com/calculus/sequences/problems_limit.html

Vectors

- http://math-exercises.com/analytical-geometry/analytic-geometry-of-the-straight-line-and-plan
- http://math-exercises.com/analytical-geometry/vectors
- http://math-exercises.com/analytical-geometry/relative-position-distance-and-deviation-betwee

They are not exactly the same as the homework exercises, but it can be useful.

4 Appendix

Here stand some common abbrevations and notations used in mathematics.

Abbrevations

appendix supplement, addendum, postscript

iff *if and only if*, also denoted by $\Leftrightarrow$

s.t. *such that*, something with the property that ...

i.e. *id est*, latin for *"that is"*

q.e.d. *quod erat demonstrandum*, latin for *"which is what had to be proven"*, marked as $\square$

e.g. *exempli gratia*, latin for *"for example"*

Mathematical symbols

$\angle$ *angle*

$\forall$ *for all*, every single one

$\exists$ *exists*, there is one

$\exists!$ *exists uniquely*, there is exactly one, one and no more

$:=$ *define equals*, the left-hand side is a new symbol or value, the right-hand side is a known thing which is the definition of the new thing.

$|\bullet|$ *absolute value* of a number

$\|\bullet\|$ *norm* or *length* of a vector

$\lightning$ this symbol marks contradiction.

The usual sets of numbers

$\mathbb{N}$ natural numbers: $\{0, 1, 2, \ldots\}$

$\mathbb{Z}$ integers: $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$

$\mathbb{Q}$ rational numbers

$\mathbb{R}$ real numbers

$\mathbb{C}$ complex numbers

Basic theorems

Theorem 4.1 (Triangle inequality)**.** For every real numbers $a, b \in \mathbb{R}$ the following holds.

$$|a + b| \leq |a| + |b|$$

In particular, for $-b$:

$$|a - b| = |a + (-b)| \leq |a| + |-b| = |a| + |b|$$

This is also true for any vectors $\mathbf{a}, \mathbf{b}$:

$$\|\mathbf{a} + \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|$$