Mathematics A1a practive

Academic year 2016 autumn

The day of November 17 is off!

See the academic calendar.

Herein I give you examples and solutions for practicing. Many of these can be found in the given parts of Thomas' Calculus. Mind that we have not covered Example 5. on class, but we will!

Example 1.

$$\int \sqrt{2^x}\,\mathrm{d}x = ?$$

Solution.

$$\int \sqrt{2^x}\,\mathrm{d}x = \int 2^{\frac{x}{2}}\,\mathrm{d}x = \int e^{\log(2)\cdot\frac{x}{2}}\,\mathrm{d}x =$$

$$= \int e^{\frac{\log(2)}{2}\cdot x}\,\mathrm{d}x = \frac{1}{\frac{\log(2)}{2}}\cdot e^{\frac{\log(2)}{2}\cdot x} + c =$$

$$= \frac{2}{\log(2)}\cdot\sqrt{2^x} + c$$

□

Example 2.

$$\int \sqrt[3]{1-\frac{x}{2}}\,\mathrm{d}x = ?$$

Solution.

$$\int \sqrt[3]{1-\frac{x}{2}}\,\mathrm{d}x = \int \left(1-\frac{1}{2}\cdot x\right)^{\frac{1}{3}}\,\mathrm{d}x =$$

$$\frac{1}{-\frac{1}{2}}\cdot\frac{\left(1-\frac{1}{2}\cdot x\right)^{\frac{4}{3}}}{\frac{4}{3}} + c = -\frac{3}{2}\cdot\left(1-\frac{x}{2}\right)^{\frac{4}{3}} + c$$

□

Example 3.

$$\int \frac{1}{x^2+2x+2}\,\mathrm{d}x = ?$$

Solution. The denominator has no real solutions: $D = b^2-4ac = 4-8 < 0$. Therefore we have to be able to transform it into

$$\frac{1}{x^2+2x+2} \approx \frac{1}{1+x^2}$$

Let's do this by completing the squares.

$$\frac{1}{x^2+2x+2} = \frac{1}{(x+1)^2+1}$$

And the latter is the linear substitution of $\frac{1}{1+x^2}$ by $x \mapsto x+1$. The linear coefficients are $a=1, b=1$. Using that:

$$\int \frac{1}{x^2+2x+2}\,\mathrm{d}x = \int \frac{1}{1+(x+1)^2}\,\mathrm{d}x = \frac{1}{1}\arctan(x+1) + c$$

□

Example 4.

$$\int \frac{1}{2x^2-3x+1}\,\mathrm{d}x = ?$$

Solution. The denominator has two real solutions: $D = b^2 - 4ac = 9 - 8 > 0$. This gives us that $2x^2 - 3x + 1 = (x-1)(2x-1)$. Therefore we have to be able to transform it into

$$\frac{1}{2x^2 - 3x + 1} = \frac{A}{x-1} + \frac{B}{2x-1}$$

with partial fraction decomposition.

$$\begin{aligned}
\frac{1}{2x^2 - 3x + 1} &= \frac{A}{x-1} + \frac{B}{2x-1} && / \cdot (x-1)(2x-1) \\
1 &= A \cdot (2x-1) + B \cdot (x-1) && /\text{expand} \\
1 &= 2A \cdot x - A + B \cdot x - B && /\text{group} \\
1 &= \underbrace{(2A + B)}_{0} \cdot x + \underbrace{(-A - B)}_{1} \cdot 1
\end{aligned}$$

Solve $2A + B = 0$ and $-A - B = 1$, this gives $A = 1, B = -2$. So

$$\int \frac{1}{2x^2 - 3x + 1} \,\mathrm{d}x = \int \frac{1}{x-1} + \frac{-2}{2x-1} \,\mathrm{d}x$$

And they are the linear substitutions of $\frac{1}{x}$.

$$\int \underbrace{\frac{1}{x-1}}_{a=1,b=-1} + \underbrace{\frac{-2}{2x-1}}_{a=2,b=-1} \,\mathrm{d}x = \log(x-1) - 2 \cdot \frac{1}{2} \log(2x-1) + c$$

□

Example 5.

$$\int \frac{1}{2x^2 - 6x + 4.5} \,\mathrm{d}x = ?$$

Solution. The denominator has one real solution, but twice: $D = b^2 - 4ac = 36 - 4 \cdot 2 \cdot \frac{9}{2} = 0$. This gives us that $2x^2 - 6x + 4.5 = 2 \cdot \left(x - \frac{3}{2}\right)^2$. Therefore we have:

$$\int \frac{1}{2x^2 - 6x + 4.5} \,\mathrm{d}x = \int \frac{1}{2} \cdot \left(x - \frac{3}{2}\right)^{-2} \,\mathrm{d}x =$$

$$\frac{1}{2} \cdot \frac{\left(x - \frac{3}{2}\right)^{-1}}{-1} + c = \frac{-1}{2x - 3} + c$$

□

Example 6.

$$\int \cos^2(x) \,\mathrm{d}x = ?$$

Solution.

$$\int \cos^2(x) \,\mathrm{d}x = \int \frac{1 + \cos(2x)}{2} \,\mathrm{d}x = \int \frac{1}{2} + \frac{1}{2} \cdot \cos(2x) \,\mathrm{d}x =$$

$$= \frac{1}{2} \cdot x + \frac{1}{4} \cdot \sin(2x) + c$$

□

Example 7.

$$\int \sin(x) \cdot \cos(x) \,\mathrm{d}x = ?$$

Solution.

$$\int \sin(x) \cdot \cos(x) \,\mathrm{d}x = \int \frac{1}{2} \cdot \sin(2x) \,\mathrm{d}x =$$

$$-\frac{1}{4} \cdot \cos(2x) + c$$

□