Math A1a - Calculus **Name (Print):** _____
Civil Engineering **Neptun Code:** _____
Final exam (Autumn 2016) **Time Limit:** 120 Minutes

This exam contains 14 pages (including this cover page) and 6 questions. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your neptun code on the top of every page, in case the pages become separated.

You are required to show your work on each question. You can collect points from any question, no restriction on your choice. You can collect at most 100 points in the whole exam.

The following rules apply:

- You must **not** use your books, notes, or any other papers. You can **only** use blue or black pen, answers written with red pen, green pen or pencil will not be considered.
- You may use your calculator but you are **not** allowed to use any other device, e.g. **YOUR PHONE**.
- Organize your work, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- Mysterious or unsupported answers will not receive full credit. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- You have to work alone, any communications with the others will cause termination of your exam immediatly.
- If you leave the room during the exam for any reason, then your exam will be terminated.
- If you need more space, ask the examiner to provide some blank papers.

Question	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
Total:	100	

Do not write in the table on the right.

With best wishes,
Dr. Mohamed Khaled and Gábor Borbély
December 20, 2016

Question 1 ..... *20 points*

(a) [5 points] Give a complex number for which $(1 - i)$ is a fourth root, then give one of the other fourth roots.

(b) [5 points] Find the fourth roots of the complex number $z = 2 - 2i$.

(c) [5 points] Does there exist a function f which satisfies $f(x^2) = x + 1$ for all real numbers? If so, give such function.

(d) [5 points] Let $A = \{x \in \mathbb{R} : 1 < x < 3\}$, $B = \{x \in \mathbb{R} : 5 \leq x \leq 7\}$ and $C = \{x \in \mathbb{R} : 2 < x < 6\}$. Determine the following sets: $A \cup C$, $(A \cup B) \cap C$ and $B \cap \bar{C}$.

Question 2 ..... *20 points*

(a) [8 points] **Rolle's Theorem**: Suppose that f is a differentiable function on the interval $[a, b]$. If $f(a) = f(b)$ then there is c between a and b such that $f'(c) = 0$. State the Mean Value Theorem and show that Rolle's Theorem follows from the Mean Value Theorem.

(b) [6 points] What are the smallest and largest values that $(\sin x \sin y)$ can have if $x + y = \pi$ and if x and y are both nonnegative?

(c) [6 points] Sketch the graph of the function $y = x^3 + 2x^2$.

Question 3 ..... *20 points*

(a) i. [5 points] By finding a partition of the interval $[0, 1]$, using the function $f(x) = e^x$ and a suitable Riemann sum show that

$$\lim_{n\to\infty} \frac{1}{n}(1 + e^{\frac{1}{n}} + e^{\frac{2}{n}} + \cdots + e^{\frac{n-1}{n}}) = e - 1.$$

ii. [5 points] Find the above limit again by explicitly finding the sum $(1 + e^{\frac{1}{n}} + e^{\frac{2}{n}} + \cdots + e^{\frac{n-1}{n}})$ and using L'Hospital's rule.

(b) Evaluate the following integrals:

i. [5 points] $\displaystyle\int \frac{1}{x\sqrt{1-(\ln x)^2}}\,\mathrm{d}x.$

ii. [5 points] $\displaystyle\int \frac{\sin^3 x}{\cos^2 x}\,\mathrm{d}x.$

Question 4 ..... *20 points*

(a) [5 points] Prove that

$$\int x^n e^x \, \mathrm{d}x = x^n e^x - n \int x^{n-1} e^x \, \mathrm{d}x, \qquad n \geq 1. \tag{1}$$

(b) [5 points] Use equation (1) above to evaluate $\int x^2 e^x \, \mathrm{d}x$.

(c) Evaluate the following limits:

i. [5 points] $\lim_{x\to\infty} \sqrt{x}(\sqrt{x+3} - \sqrt{x-2})$.

ii. [5 points] $\lim_{x\to 2} \dfrac{x-2}{x^3-8}$.

Question 5 ..... *20 points*

(a) [5 points] Find the relative position, the angle and the intersection (if any) of the lines $(3,1,3)+(1,0,1)t$ and $(2,2,0)+(1,1,-1)t$.

(b) [5 points] How do you determine the volume of a Tetrahedron (3D)? Suppose that you are given the coordinates of each vertex.

(c) [5 points] $\lim_{n\to\infty} \frac{\cos(n)}{\sqrt{n^2+1}} = ?$

(d) [5 points] Is the following sequence bounded from below, above, neither or both?

$$\cos(n \cdot \pi) \cdot n$$

Question 6 ..... *20 points*

(a) [5 points] Differentiate the following function $y = \cos(x) \cdot e^{\sin(x)}$.

(b) [5 points] Determine the tangent and normal of the graph $y = \sqrt{1 - x^2}$ at $x_0 = \frac{1}{\sqrt{2}}$.

(c) [5 points] Determine the center of the osculating circle of $\arctan(x)$ at $x_0 = 1$.

(d) [5 points] Evaluate the following improper integral:

$$\int_e^\infty \frac{1}{x \ln x} \, \mathrm{d}x.$$

Sum Formulas

- $\sum_{k=1}^{n} c = nc$
- $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$
- $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.
- $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$.
- $\sum_{k=1}^{n} r^{k-1} = \sum_{k=0}^{n-1} r^k = \frac{1-r^n}{1-r}$ if $r \neq 1$.