The Russell Operator

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> Context • The question of how to understand the epistemology of set theory has been a longstanding problem in the foundations of mathematics since Cantor formulated the theory in the 19th century, and particularly since Bertrand Russell articulated his paradox in the early twentieth century. The theory of types pioneered by Russell and Whitehead was simplified by mathematicians to a single distinction between sets and classes. The question of the meaning of this distinction and its necessity still remains open. **> Problem** • I am concerned with the meaning of the set/class distinction and I wish to show that it arises naturally due to the nature of the sort of distinctions that sets create. **> Method** • The method of the paper is to discuss first the Russell paradox and the arguments of Cantor that preceded it. Then we point out that the Russell set of all sets that are not members of themselves can be replaced by the Russell operator R, which is applied to a set S to form $R(S)$, the set of all sets in S that are not members of themselves. **> Results** • The key point about $R(S)$ is that it is well-defined in terms of S, and $R(S)$ cannot be a member of S. Thus any set, even the simplest one, is incomplete. This provides the solution to the problem that I have posed. It shows that the distinction between sets and classes is natural and necessary. **> Implications** • While we have shown that the distinction between sets and classes is natural and necessary, this can only be the beginning from the point of view of epistemology. It is we who will create further distinctions. And it is up to us to maintain these distinctions, or to allow them to coalesce. **> Constructivist content** • I argue in favor of a constructivist perspective for set theory, mathematics, and the way these structures fit into our natural language and constructed speech and worlds. That is the point of this paper. It is only in the reach for absolutes, ignoring the fact that we are the authors of these structures, that the paradoxes arise. **> Key words** • **Distinction, Russell paradox, Barber paradox, sets, Cantor's Theorem, classes, types.**

1. Introduction

This is a paper about how a universe comes into being through the making of a distinction (Spencer-Brown 1969). But rather than starting from nothing and noting that the set theoretic universe arises from the act of distinction that creates sets, we shall concentrate here on the Russell paradox about the set of all sets that are not members of themselves. We shall examine the distinctions that are made to keep the paradox at bay. There is a curious history here, with Bertrand Russell and Alfred North Whitehead creating a complicated theory of types that is later made into a single distinction between sets and classes. This class/set distinction appears to suffice for mathematical purposes, but we argue that it, as any distinction will, spawns a universe of distinctions and structures just like the theory of types, once one attends to the epistemology of the mathematics in relation to language and thought. Then we see that the solution to these conundrums lies not in the formalisms, but in our ability to create formalisms, in our ability to create and handle distinctions. These worlds made of distinctions are imaginary and fragile, yet they are often strong and real. They are our creations and our ability to make them forms the base of all that we do.

2. The Russell Paradox

The purpose of this paper is to discuss the Russell Paradox.

Lets begin with the Barber.

The Barber: There was a town, long ago, wherein lived a barber, and he shaved those and only those who did not shave themselves. The question is – who shaves the barber?

If the Barber shaves himself then he shaves someone who shaves himself and this is not allowed. If the Barber does not shave himself, then he must shave himself, since the barber shaves everyone who does not shave himself.

What are we to do with this dilemma?

It is well-known that the Barber reappears in the Russell Paradox as the set R of all sets x that are not members of themselves. Is R a member of itself? If so, then it cannot be a member of itself, and if R is not a member of itself then it must be a member of R. Here the problem is suddenly of serious intellectual import. It was assumed by Gottlob Frege and others that to each concept there should be the set of those entities that satisfied the concept. The Barber in the form of the Russell set shows that this cannot be done without some sort of control.

What control should be imposed'? Of what should we be suspicious in examining Russell's Barber? There are two suspects. One is the use of "all" and the other is the notion of self-reference, of self-membership. Russell apparently suspected them both, but had more suspicion of the matter of self-membership than he did of the use of the "all." He created a solution (the theory of types) that has a lot of control built in; too much for most working mathematicians.

And so the problem underwent an evolution that eventually led to a very simplified theory of types for mathematicians (Kelley 1955), where there were sets and there were classes. *Both sets and classes are meant to designate certain collections, but a set must* *be a member of a class in order to be a set.* In this evocation of pure mathematics, every set has members that are sets.

You never find anything but sets when you look inside a set. Of course there is the empty set { } and then you find nothing at all.

If x is a member of a class then it is a set. Thus sets are members of classes, and classes are certain collections of sets, but classes are not members of anything! It is all right for most mathematics to have it this way, but if you wish to speak of "All Ideas" then this collection is itself an idea and it is not at all obvious when an idea is a set.

We now take R to be *the class of all sets that are not members of themselves.* If R were a set then we would get a contradiction. Thus R is not a set. R is a class and we are done with the problem. This is the Hilbert-Gödel-Bernays solution to the Russell paradox.

Some concepts have as their extension a class and some concepts have as their extension a set. Some collections are classes but not sets.

How would this resolution apply to the Barber? Barbers exist and practice their profession. So the Barber of our tale is apparently wrongly defined. He has a bad job description.

Perhaps there are sets like that, sets that cannot accept a given job description. Georg Cantor had, some time before Russell and the paradox, pointed out just how some sets could not support a natural enough condition. Cantor proved the following theorem.

Cantor's Theorem. There cannot be a 1–1 correspondence between a set U and the set of its subsets 2^U.

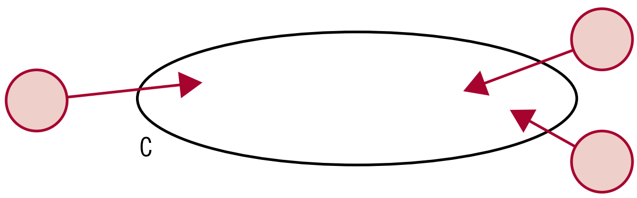
Proof. Suppose that there were such a 1–1 correspondence. Then we could have a unique element x of U associated with every subset X of U. Think of each X as equipped with a pointer. This pointer will point to an element of the set U. Now each X is a subset of U and so X will either point to one of its own members or X will point to some x that is not a member of X. In the latter case we shall say that "X points outward" and in the former case we shall say that "X points inward." Form the following subset C: *C is the set of all x in U such that x is the target of an outward pointing subset X of U.* Does C point outward or does C point inward? If C points inward with target x, then x must be a member of C, but that would mean that C points outward since all elements of C are outward target points. If C points outward, then x would not be a member of C but if C points outward then x must be a member of C. We have a contradiction. The only way out is to conclude that there cannot be such a 1–1 correspondence, and this proves the Theorem. //

BOX 1: A Cantorian tale

There was a set *U*.

And it was said that every subset of *U* would be equipped with a pointer that pointed to an element of *U*, unique to that subset. Some subsets pointed outside themselves. Some subsets pointed to one of their own members.

Tales of great battle and heroism were told about these subsets. One day the Storyteller was confronted by a strange bearded fellow who called himself Cantor. Cantor said, "Tell me a tale about the set of all points in *U* that are at the tips of outward pointers from the other subsets of *U*!" "Of course I am happy to tell you of that set, and I shall call it *C* for Cantor," said the Storyteller. "Your set looks like this."



"All the sets with outward pointers point into your set *C*." And Cantor then asked, "But what about *my* pointer? Which way does my set point?" And suddenly the Storyteller was silent and thoughtful. And he said, "Oh my. If your set points inward then it must be an outward pointer. But if your set points outward, beyond itself, then by its very definition it must point into itself. Alas, I am so sorry, your set cannot have a pointer, and I was wrong. It is not possible for every set to have a pointer as I had imagined." And Cantor was not sure whether this turn of events pointed to sadness or to happiness.

Another way to put this argument is to suppose that only *some* of the subsets of U have pointers. Then C shows that there will always be subsets that cannot point. C is just like the Barber. C serves up those subsets (i.e., their targets) that point outwards. But in the process, C gets entangled with serving up itself. C is just the Barber in disguise, and in this situation we have to conclude that the Barber cannot do the job to which he was presumably assigned. Cantor's Theorem is very important, simple as it is, for it shows us that the set of subsets of a set is always larger than the set itself. This means that there is an ever increasing hierarchy of infinities starting with the first infinite set $N = \{1, 2, 3, \ldots\}$ of natural numbers.

Then we have $N < 2^N = N' < 2^{N'} < \ldots$ with no limit to the size of the infinite sets so generated.

Remark. See Box 1 for a illustration of this argument in the form a small fable.

Let us look at this more closely. Let X be *any* set. Let $R(X)$ denote the set of sets x in X such that x is not a member of itself.

$R(X) = \{x \mid x$ is a member of X and x is not a member of itself.$\}$

We did not take all sets. We just chose a particular set X and formed $R(X)$. The Russell argument still applies, and if $R(X)$ were a member of X then we would get a contradiction. So $R(X)$ cannot be a member of X. We have shown that *sets are always incomplete.* There is no such set as the set of all sets. If X were the set of all sets we get a contradiction. There is the *class* Ω of all sets. That is OK but there is no set of all sets. Every set is incomplete and given a set X, $R(X)$ is a new set that is not a member of X.

Let us go back to the Barber. We could have told the story differently. We could have said, "There is a Barber and he shaves everyone in the village of Königsberg who does not shave himself." And now you see a way that such a barber can exist. He simply is not an inhabitant of Königsberg. This is the set theorist's solution to the Barber paradox.

By the way, Ω is a very nice class. After all, if S is any set of sets, then S is itself an element of Ω. So $\Omega = 2^{\Omega}$. Ω is identical with its class of subsets. But Ω is a class that is not a set.

We can try to get a contradiction from the assumption that Ω is a class by saying that every element x of Ω is a subset of Ω. And so we can have every element x point to *itself.* Then the elements of Ω divide into those that are members of themselves and those that are not members of themselves. In this case, Cantor's collection C is the collection of all x in Ω such that x is not a member of x. But we have seen that C is not a set! So C is not a member of Ω (members of classes are always sets) and there is no contradiction. Only sets can be members of a class.

Are we done? I am afraid not. Not if you want to work in the languages that are spoken in the world. Let me tender persuasions. Consider the collection of games – two-person games such as checkers, chess and tennis. We will be particularly interested in games that are *finite* in that they end in a finite amount of time. Consider all such games, and let us define a new game that we shall call "Hypergame." Here is how to play Hypergame.

1 | The first player says, "Let's play Hypergame."
2 | The second player says, "Let's play ____." (where ____ is a finite game of the second player's choice)

The two players then proceed to play a round of ____.

So as you can see, every round of Hypergame ends in a finite amount of time since the players always choose a finite game to play. Thus *we have proved that Hypergame is a finite game.*

But since Hypergame is a finite game we can have the following exchange:

1 | Let's play Hypergame.
2 | Let's play Hypergame. (Now player number 2 is in the first position)
1 | Let's play Hypergame.
2 | Let's play Hypergame.
…

and this (rather silly) round of Hypergame goes on forever.

So Hypergame is not a finite game. This is a contradiction.

The only known way out of this problem is to declare that Hypergame is not an ordinary game and cannot be allowed as a choice when playing Hypergame. But surely Hypergame is a game. Would you like to play?

It seems that there is a black hole at the center of logic and we just have to accept that and live in a world created by the distinctions we form to have a world at all.

In the rest of the paper we will explore the role of making distinctions in mathematics.

3. The class/set distinction and the Russell Operator

If we look back at the logic by which we have avoided the Russell paradox by making the class/set distinction, it seems both artificial and natural. Let us look at it again. In the theory of sets there are entities that are sets and entities that are classes. An entity x can be a member of an entity y. But classes can not be members. Sets are characterized by being members. Thus if x is a member of y then x is a set. Classes are peculiarly distinguished by not being members of anything. This works just right, it seems, to avoid the paradoxes, but leaves us feeling a bit odd about the classes. Let us consider Ω, the class of all sets. If x is a set then x is a member of Ω.

You might think that Ω would be paradoxical, but no problem arises. We made this argument in the previous section, but let is repeat it now.

We map $I: \Omega \to 2^{\Omega}$ by $I(x) = x$. This is a legitimate mapping because every set is a set of sets and hence is an element in the class of subsets of Ω. Then we try to apply Cantor's argument and we say, "Form the class $C = \{x \text{ in } \Omega \mid x \text{ is not a member of } I(x)\}$." The usual argument is that C is a subset of Ω and that C cannot equal any $I(a)$ for any a since it differs from each one. But here we have $I(x) = x$ and so $C = \{x \text{ in } \Omega \mid x \text{ is not a member of } x\}$. Thus C is the Russell class and we have already ousted C from being a set. Thus C is not a member of Ω and there is no contradiction!

We have already argued that this solution is really not so artificial. The Barber could be from another town. The Russell set could be a new sort of set. The Hypergame could be a new sort of game.

We are asked to understand that new distinctions have to be made in order to avoid circularity and contradiction. Sets are entities for mathematical construction and calculation. Ω, the Class of all Sets, is available to us. Yet we find that some entities will not be members of Ω. And Ω itself cannot be the member of any class.

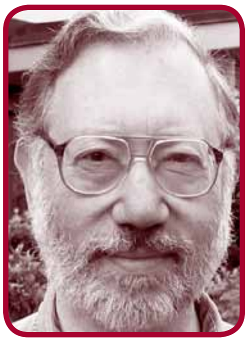
But I can conceive of a "class" whose only member is Ω. I will even write it down. There:

$$\{\Omega\}.$$

What have I done? This entity, sitting on the page, represents neither a set nor a class. Its meaning is clear. It represents my thought in collecting together just Ω. But if $\{\Omega\}$ is a mere class, then Ω must be a set. It is not a set and so $\{\Omega\}$ is a hyperclass, and $\{\{\Omega\}\}$ is a hyperhyperclass and $\{\{\{\Omega\}\}\}$ is a hyperhyperhyperclass and so on ad infinitum.

If you wish to have such things you will apparently have to create hyperclasses and hypersets, and this process will go on forever. At this point we meet the demon that Russell originally encountered in his theory of types.

The Russell Demon. We have defined the Russell operator R on sets S as $R(S)$, which is the set of all members of S that are not members of themselves. We have remarked that $R(S)$ will never be a member of S. It helps to familiarize oneself with this operator on small sets. For example, if $\{\,\}$ denotes the empty set, the $R(\{\,\})$ is empty and so $R(\{\,\}) = \{\,\}$. On the other hand, suppose that $I = \{I\}$ is a set that has only itself as a member. Then we have $R(I) = \{\,\}$ and we



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see that $R(I)$ is not a member of I. We now have two sets and can form $S=\{I, \{\ \}\}$. Then $R(S)=\{\{\ \}\}$ and a new set has come into being. Collecting all the sets so far, we have $T=\{I, \{\ \}, \{\{\ \}\}\}$ and $R(T)=\{\{\ \}, \{\{\ \}\}\}$. This is the next new set. We can define a recursive process by letting $X'=X \cup \{R(X)\}$ for each set X. Then we have $I'=S$, $I''=T$ and so on. We can regard the simple set that is its own member as the initial point in the generation of an infinite hierarchy of sets. It is significant that the Russell operator creates the empty set *from* the self-referential set. This is a formal image of the theme of this paper, that even the empty set is a creation. Taking the Russell operator out to the class Ω, we can try to form $R(\Omega)$. But $R(\Omega)$ is the original Russell class. It is not a set. Thus $R(\Omega)$ is a class and does not belong to Ω. Thus even at the level of classes, the Russell operator continues to construct new entities. If we start collecting up the so-created classes into superclasses, the process will start all over again. The demon never sleeps.

4. Conclusion

It only takes the one distinction, class/set, and eyes-wide-open epistemology to see that the one distinction opens the door for Russell's demon and we are off and running with a theory of types, starting at the level of the class of all sets. It only takes one distinction to create a whole universe.

Mathematicians have been content with sets and an occasional tip of the hat to classes, but from the point of view of epistemology and philosophy, one must face up to the Russell demon. We cannot discard our concept of classes. But this concept is necessarily a hyperconcept. Oh my.

There are no hyperconcepts. Only concepts. And we handle them just fine. It is not just misplaced confidence. It is our ability to handle entities and frameworks equally and to keep making sense out our own created world.

But how do we do that? That is just it, *we do that.* It is our creation.

We do that by taking the freedom to make and unmake distinctions as they are significant to us, to track down contradictions or to use circularities as they are useful for us. Any formality in which a contradiction can arise is built from distinctions that we have made.

When we go to the level at which these houses of cards are built and take ownership of the construction, then the world of logic comes aright.

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