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Small-group searches for mathematical proofs and individual reconstructions of mathematical concepts

Draga Vidakovic[a,*], William O. Martin[b]

[a] *Department of Mathematics and Statistics, Georgia State University, 30 Pryor Street, SW STE 750, Atlanta, GA 30303-3083, USA*
[b] *North Dakota State University, USA*

Abstract

This is a study of mathematics students working in small groups. Our research methodology allows us to examine how individual ideas develop in a social context. The research perspective used in this study is based on a *co-constructive* view of learning. Groups of three or four undergraduate mathematics majors, with prior experience writing mathematical proofs together, were asked to prove three statements. Computer software, such as *Geometers Sketchpad*, was available. Group work sessions were videotaped. Later, individuals viewed segments of the group video and were asked to reflect on group activities. Students in some groups did not share a common conception of proof, which seemed to hamper their collaboration. We observed interactions that fit with the co-constructive theory, with bidirectional interactions that shaped both group and individual conceptions of the tasks. These changes in understanding may result from parallel and successive *internalization* and *externalization* of ideas by individuals in a social context.

Keywords: Small-group work; Mathematical proof; Co-constructivism; Group interactions; Technology environment; Learning of mathematics; Learning of geometry

1. Small-group work in mathematics

There is no shortage of suggestions for improving the mathematical learning that takes place in college classrooms. Since the mid-1980s especially, increasing numbers of mathematicians and collegiate

* Corresponding author. Tel.: +1 404 651 0648.
E-mail address: dvidakovic@gsu.edu (D. Vidakovic).

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mathematics educators have devoted considerable attention to the nature of undergraduate mathematics courses. They have focused on each of the core areas of content, instruction, and learning, producing reformed curricular materials and instructional strategies, and developed, tested, and revised theories of learning.

Work in small groups is one form of instruction that has been given increased attention in mathematics classrooms at the collegiate level during the past decade (Johnson, Johnson, & Smith, 1991a, 1991b; Rogers, Reynolds, Davidson, & Thomas, 2001; Treisman, 1992). Constructivist learning theories and research provide support for the notion that cooperative and collaborative problem solving can be an effective aid in the teaching and learning of mathematics (Davidson, 1990; Davidson, & Kroll, 1991; Johnson et al., 1991b; Reynolds et al., 1995; Vidakovic, 1997). Undergraduate mathematics instructors now can find many resources that encourage and help them to incorporate cooperative and collaborative instructional methods in their courses (Dubinsky & Mathews, 1997; Reynolds et al., 1995). Underlying a move to small-group work is a belief that when students discuss their thinking about problems with others it helps them develop rich and powerful understanding of mathematical concepts, perhaps by a structural organization and connection of mental constructs (Hiebert & Carpenter, 1992). Any teacher or tutor of mathematics has experienced the increased understanding of a topic that comes, apparently, from the act of explaining mathematics to others. Such beliefs and experiences support the idea that effective cooperative group work on appropriate mathematical tasks can be a highly effective instructional strategy.

We have students work in small groups in our own undergraduate classes for similar reasons. While our experience and course assessment supports our beliefs about the benefits of small-group work, we feel driven to more systematic investigations to better understand the nature of group interactions and their apparent impact on learning.

Initially, we have chosen to focus on the construction of proofs by undergraduate mathematics students. We chose this particular domain for several reasons: (a) the central role of proof in mathematics; (b) constructing proofs is a high-level problem-solving task that requires both factual recall and original thinking; and (c) even good mathematics students are known to have difficulty developing their ability to understand and construct mathematics proofs (Selden, Mason, & Selden, 1989; Selden, Selden, & Mason, 1994). Observations of individuals' actions while they struggle with a familiar or unfamiliar problem can provide considerable insights to their thinking, attitudes, and beliefs about mathematics.

The objective of our research is to initiate an exploration of the process of individual learning in the social context provided by small-group problem solving situations. Specifically, in this study we seek to analyze group's decisions and individual thinking as upper-division undergraduate mathematics students try to prove elementary theorems of Euclidean geometry. We believe that better understanding of these individual and group processes will help us understand the nature of learning in small-groups and will provide guidance for enhancing the learning in undergraduate mathematics classes. Hershkowitz (1999) expressed the need to focus on the "individual construction of knowledge within the different 'ensembles' of which he or she is part."

The literature suggests that a problem-solving environment that promotes rich, social interactions around the material increases the likelihood of students' individual learning (Denning & Smith, 1995). Technologies, including computers and computer laboratories, are an example of instructional settings that have been used to provide especially rich environments to enhance students' interactions by some collegiate mathematics educators (Dubinsky, 1995; Harvey, Waits, & Demana, 1995; Hillel, Lee, Laborde, & Linchevski, 1992; Judson, 1990; Kaput & Thompson, 1994; Shaw, Jean & Peck, 1997). The Geometers Sketchpad (Key Curriculum Press, 2004) is a specific dynamic software package that was used in small-group settings by the participants of this study during an undergraduate geometry course to investigate geometric problems. We believe that the richness of interactions that occurred in some of the groups, as we will describe later, partly reflected the environment provided by such software. Students in our study used the Geometers Sketchpad (GSP) in their geometry courses and the GSP was available for their use during the research interviews. We wish to emphasize that individual mathematical thinking in a social context is the primary focus of this study. Nevertheless, the students' use of technology during their geometry course and in the interviews clearly influenced their thinking, as will become apparent in our description and analysis of the data.

2. Theoretical context

Since the middle of the 20th Century, two distinct perspectives on learning have predominated in much of the mathematics education research. One is a psychological focus on individual learning and knowledge. Important examples include Piaget's developmental psychology, which has influenced much recent work in mathematics education research (for example, the APOS theory of mathematical learning described by Asiala et al., 1996). The psychology of mathematics learning was an important research domain during the second half of the 20th Century. Cognitive scientists focused on individual mental processes in a move beyond a behavioristic view of mathematics learning. Along with developmental theories in the style of Piaget, there was also a focus on information-processing as a model of the activity of the individual mind during learning (see Resnick & Ford, 1981, for an earlier survey of the psychology of mathematics).

The second perspective is on the social nature of knowledge and learning. Vygotsky's (1962) work on language and meaning has influenced later work in mathematics education such as socioconstructivism, where knowledge and reality are viewed as social, rather than individual phenomena (see, for example, Ernest, 1991; von Glasersfeld, 1990). Lakatos' writing (1976) about the nature and development of mathematics knowledge is a similar example of this perspective on knowledge as a social construct, rather than something that has independent or absolute existence.

Our study is not as philosophical as the perspectives just mentioned, but we are very interested in the possibility of relating individual and social notions of knowledge. Researchers from both perspectives recognize the importance of social activities for much mathematical learning. Valsiner (1992, 1993) has proposed a theoretical perspective, *co-constructivism*, that we find useful for describing how individual learning takes place in a social setting.

2.1. Learning as a social activity: small-groups environment

We use the term *small-group* work to represent a combination of collaborative and cooperative work. Small-group problem solving is socially organized activity. We view students' collaboration as based on a shared conception of the task.

Students bring pre-existing schemas, from academic and from other life experiences, to small-group settings. We classify as small-group problem solving only those group activities in which two or more individuals are cooperating to ensure their own learning and facilitate the learning of all others in their group. We believe that small-group problem solving occurs when the shared knowledge stays within the individual *zones of proximal development* of group members. Briefly, the *zone of proximal development* is defined as the difference between the level of an individual's actual development and more advanced level of potential development that could be observed in interaction between more or less capable participants (Vygotsky, 1962).

2.2. Learning as individual development in a social context: co-construction of knowledge

We have adapted Valsiner's notion of *co-constructivism* (Valsiner, 1992, 1993) as the theoretical framework for our research. This perspective synthesizes the ideas of a number of developmental theorists—Piaget, Vygotsky, Stern, Wertsch—all of whom have been fascinated by the ways in which persons develop as both individual and social entities. Co-constructivists view learning as the joint construction (or *co-construction*) of the psychological system of the developing person by him/herself, and the "social others" who influence the development of the individual psychological framework through attempts to communicate ideas. Learning is seen as arising from the two-way interplay between individual and social activities.

The co-constructivist view blends the complementary constructivist and sociogenetic viewpoints in the learning process. Valsiner (1987) recognized that there is (a) a collective culture of socially shared meanings and (b) the individual's personal cultures. Thus, culture is partially shared and partially personal. Because of personal contributions, individuals are said to *co-construct* the collective culture.

According to Valsiner, individuals construct their personal meanings from the collective cultures by way of *internalization*, while at the same time contributing to the reconstruction of that collective culture by process of *externalization*. Although one constructs her/his knowledge socially—through negotiation of meanings, in Vygotsky's language (1962)—Valsiner emphasizes that because of individual experiences, it is unlikely that two people construct exactly the same understandings. Since psychological development is an open-systemic phenomenon in which *novelty* is constantly in the process of being created (Valsiner, 1987, 1989a, 1989b, 1991), reality is characterized as a dynamic phenomenon as it moves between individuals and collective individuals. In a group setting, the group or collective knowledge is constructed through negotiation. When the process of negotiation results in agreement, that agreement is reality or social reality.

From the standpoint of an observer, we could say that the development of a group's understanding of a mathematical idea, both collectively and at the level of individuals in the group, consists of a series of internalization and externalization transformations or representations alternating with one another. The notion of internalization implies a critical transition or transformation from perceived external social experiences to individual inner thinking, invoking new mental functions within the individual. The formation of new mental functions takes into account the individual's previous experience, their mental structure, and the dynamic nature of group interactions. In parallel with transformation of external experiences to the internal sphere *internalization*, the reverse process, occurs. That is, the process of transformation of internal experience into the external expression, *externalization*, takes place completing a cycle and making it possible to study the cognitive development of an individual. We can infer these changes by comparing the original, external expression of ideas with the transformed expression that follows an internalization and externalization cycle. Externalization is a constructive process. According to Valsiner (1993), externalization involves constructive transformation of the internalized psychological phenomena into the social, interpersonal domain.

Valsiner suggested the existence of two forms of coordination between internalization and externalization: (a) parallel functioning and (b) delayed functioning of the externalization with respect to internalization. Any externalization feeds back some internalization, which is the source of new externalization, and the cycle continues. For example, when we, as teachers, provide some information to our students we are only asking them to internalize. If, at the same time, we ask them to reflect and share their thinking with others we are encouraging the externalization of their thoughts. Social activities such as class discussion may produce a new series of internalization and externalization transformations of ideas. We expect this increased student activity or engagement to facilitate *learning*; that is, changes in their psychological framework.

In teacher-centered classrooms students are not asked to reflect and share immediately; to internalize, then externalize ideas. More commonly, the process of externalization is left until a subsequent assessment (for example, assignment, quiz, or test). It is quite possible that by delaying student's externalization he or she misses the opportunity for better internal transformation and consequently further development of understanding relative to the individual's prior knowledge.

Valsiner's view of learning seems to differ from other theories in another important way. Rather than seeing internalization and externalization as inverse, hence reversible, functional operations that are determined by existing conditions, he instead describes an uncertainty principle he calls *canalization*: "A set of constraints that direct—but do not precisely determine the next state of human conduct." (1993, p. 25)

The co-constructivist perspective on human development is based on the general view on development by way of the principle of "bounded indeterminacy" (Valsiner, 1989, for criticism, see Van Oers, 1988). By the use of constraining as a process that enables construction of novelty (Winegar, 1988; Winegar, Renninger, & Valsiner, 1989), it is possible to explain the directionality of development, while retaining the open-systemic notion of unpredictability of the exact outcomes.

We cannot be certain of the form of the internalized notion that an individual holds, but can only make inferences based on externalized representations. Furthermore, because these processes are not completely deterministic or predictable—by the principle of bounded indeterminacy—we cannot even expect externalizations to be consistent from one expression to another, even under comparable conditions. But we can be sure that individual internalizations and externalizations, although inconsistent and unpredictable, are refined in the process of group interactions.

This co-constructive theory suggests that we can expect several things when observing small-group problem solving activities of mathematics students. Individual comments or actions may reveal changing individual mathematical ideas as the problem solving progresses, changes reflecting ideas that were expressed by other group members. For example, an individual might express a changed conception of mathematical proof that reflects group discussions. We also might see group conceptions, such as in agreed-upon responses to problems, change during the course of an interview. We might also observe the group's implicitly adopted view of proof evolve to reflect a synthesis of individual ideas communicated during the session. We do not expect to be able to give a prescription relating specific conditions particular productive group interactions or development of individual mathematical ideas, given the *bounded indeterminancy* inherent in the theory; we do expect to use our analysis to describe conditions that facilitate or inhibit communication and learning in small-group settings.

3. Research design

Stimulated recall is an introspective method in which the subjects are prompted (via appropriate stimulus such as a video- or audio-taped event, or a written document) to recall thoughts they entertained while carrying out a particular task (Gass & Mackey, 2000). The method has been in use for several decades, especially during the past 10 years, perhaps reflecting the increased use of qualitative research methods in educational studies. The method is superior to a simple post ad-hoc interview because it reduces the subject's reliance on memory without prompts.

The most basic problems of stimulated recall methodology are its reliability and validity. The assumption underlying introspective methods, which could be questioned, is that higher order cognitive processes are accessible to introspection. Retrospective reports are subject to the participant's memory limitations, especially when the considerable time intervenes between the event and the recall. Gass and Mackey (2000) provide a more complete discussion of advantages and disadvantages of the method. This methodology seemed compatible with our theoretical framework and ideally suited to our focus on the introspection of individuals in a group context, despite the potential concerns of reliability and validity.

We have used a modified version of stimulated-recall research methodology that allows us to explore student interactions in small groups (Schoenfeld, 1985). Briefly, there are three stages in this research methodology:

- Teach a course using small-group work extensively.
- After students have extensive experience with small-group course work, videotape them as they work—in the same groups as during the course—on a series of tasks related to, but not part of the course content.
- After reviewing the video of the group session, interview students individually about their experiences working in the groups—both during the course and during the videotaped interviews—using video clips of their group activities as stimuli to help trigger recollections of those problem-solving sessions.

We interviewed all our participants individually within two weeks of the group sessions to minimize concerns about memory lapses. The student's written work and transcribed protocols of the group and individual sessions provided the data for analysis.

This study, conducted at two research universities with graduate programs in mathematics, involved students studying axiomatic geometry in undergraduate mathematics and secondary mathematics education programs. All participating students had worked in small groups extensively. Also, the students had been required to write formal geometric proofs in mathematics courses completed before participating in this study. They were aware of differences between deductive and inductive proof as well of various proof procedures. The instructor rarely gave lectures in these classes, and normally only wrote proofs for students after they had spent considerable time trying to write their own proofs. Students were responsible for reading and making sense of the course text (Cederberg, 1989) and for asking questions about content with which they had difficulty: initially of their group members, then of other classmates, and only finally of the instructor. Class time was spent with students working on problems in their groups, writing solutions on the board and explaining work to the class, and having discussions about problems and material in the text. A major part of their course grade was based on homework and test problems completed by their group, which was permanent for the semester, along with some individual assignments. Students in the courses also investigated Euclidean and non-Euclidean geometries through a series of labs that used physical objects, such as balls and string, and computer software, such as the Geometers Sketchpad and NonEuclid.

Our focus in this study is on the thinking of students who had considerable experience with this collaborative style of instruction and on the interactions of these students when working on problems in small groups. This paper is specifically based on a series of interviews conducted with seven groups of 3–4 students at two universities.

Students, working in groups, were asked to write proofs of geometric statements. The problems, some of which the students could have seen in a high school geometry class, were chosen to be challenging but accessible to the students, while not being directly taken from their college geometry course. Students were asked during the interview whether they remembered having worked or seen the problems before some did. The problems were not intended to be novel tasks in the sense of those used by researchers, such as Schoenfeld (1985), who studied mathematical problem solving.

These sessions, lasting up to three hours, were videotaped. Students had access to computer and hand-held calculator technologies, such as the Geometers Sketchpad and Texas Instruments TI-92, during the problem sessions. Technologies were not required, nor suggested, for the problems and were only available in case students choose to use them. There was no explicit suggestion that technologies should be used. Two problems that most groups worked on are thoroughly analyzed in this paper: Additional questions were provided, but most groups spent all their time on these problems (see Fig. 1).

Not all groups were able to find satisfactory solutions, in their own view, during the group session. The researcher/interviewer reviewed each videotape after the session. His or her goal was to select several significant segments that appeared to involve interaction among group members that may have influenced subsequent thinking and activities of individuals.

Following the group sessions, we interviewed students individually. These individual interviews were structured around three areas:

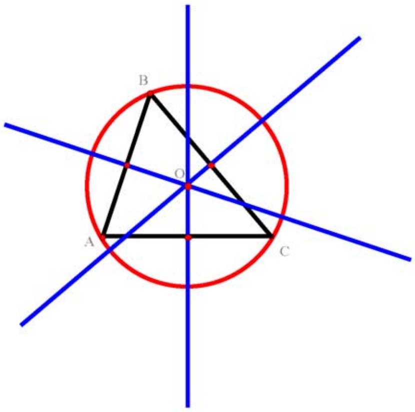
1. Each student was asked to recall parts of the problem solving session: (a) What they understood the problem to be, (b) to describe their solution, (c) to describe the contributions made by different group members, and (d) how they recalled that the solution was developed.
2. The student's feeling about group work in the geometry course: Did they believe group work helped their learning and did their group function effectively? They were specifically asked to reflect on the contribution and role of each group member during the course.
3. While viewing clips selected by the researcher, the student was asked to reflect on their own thinking and how the activities in the short segment fit within the entire problem solving session.

In this report we focus on students' responses to Question 1 and Question 3 with a brief note about their responses to the last part of Question 2. Students viewed selected segments from their own videotaped sessions and were questioned about what they observed and about their interactions with group members. This gave us the opportunity to probe each group member's thinking about the problem solving without influencing their initial activities while they worked on the problem. This also enabled us to investigate the interactions and roles that students played, or were perceived to play, in group activities. Our questions focused on the ways that group interactions appeared to influence or shape the mathematical thinking of individual students, and how individuals shaped the direction taken by the group.

Although the problems specifically requested proofs, during the interviews we did not give any guidance to students about the meanings and expectations. When students asked about our expectations, we told them to do what they believed was necessary and said we would not give any indication of whether or not their actions were appropriate. This approach allowed us to observe a variety of interpretations of proof among participants.

The context of Geometry was significant in the study only because it was linked with students' previous cooperative learning experiences and because it provided a familiar context in which students could construct formal mathematical proofs. The research method could easily be applied to other areas of mathematics or even to other disciplines.

Problem 1. In Euclidean geometry it appears that the perpendicular bisectors of a triangle are concurrent at a single point (see figure). Prove that they are and that the point is the center of a circle that contains the three vertices of the triangle (it is called the circumcircle).



Problem 2. Let ABCD be a quadrilateral in Euclidean geometry(see figure). Connect the midpoints of adjacent sides (e.g., midpoints of AB and DA) to form another quadrilateral. Name, with proof, the type of quadrilateral that is formed.

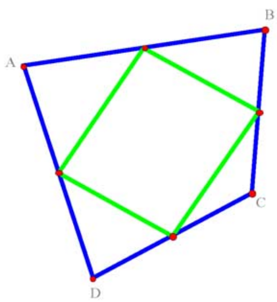


Fig. 1. Problem-solving tasks.

In contrast to some problems used by researchers specifically studying student notions of proof, we did not leave open the question of whether or not the statements students were asked to prove might be false. The tasks, then, were in keeping with what Schoenfeld (1985) said many college students believe about proofs: That the role of proof is to confirm something that is intuitively obvious, or to verify something that we already know is true. This is in contrast to a mathematician's view of proof as the search for, and development of understanding of what is true.

4. Analysis of group-work interviews

This section provides information about the data that was obtained, and includes commentary on how that data is related to the theoretical framework and some specific, interesting findings about group work and the students' understanding of mathematics. We believe the inclusion of longer excerpts here will give the reader the opportunity to judge the validity and objectivity of our interpretations based on the data obtained from the interviews.

Although several of the groups had very productive problem-solving sessions, we have chosen to present the analysis of Group 6 because their interchanges help to clarify the mechanism of co-construction. We first describe one of these productive sessions in some detail to provide insight to the sort of data produced by this study. We considered this session *productive* because students had extensive interactions that clearly influenced their thinking about the problem and because they achieved a solution that satisfied the entire group (an informal and inductive proof in which the student's warrant was purely empirical). Several other groups had similarly productive sessions that led to valid proofs of one or more statements, while some groups had much less productive sessions both in terms of lack of proof and the nature of interactions of group members. To conserve space, we comment on these other sessions without providing full details. We are unable to report our analysis of all of the data collected during this study in the limited space of this paper.

4.1. Introspection of individuals in small-group environment

The problem-solving activities described earlier provided an environment where students could engage in meaningful exploration of a problem with rich, social interactions. The extent and efficacy of interaction varied considerably from one group to another.

Our focus is on several aspects of the interactions and reflections: (a) the nature of interaction, including whether communication of ideas occurred and if this had an influence on individual thinking; (b) the proximity of individual student's thinking about the tasks and the extent to which they had a shared purpose; and (c) the students' perceptions of their own and other group members contributions to the task and mathematical understandings. The aspect (b) requires the most interpretation by the researcher; the aspects (a) and (c) are more descriptive of explicit behaviors or comments made by the students.

We observed three stages in several of the group-work interviews: (i) a preliminary *exploratory* stage as students gained familiarity with the problem; (ii) a middle *productive* stage when groups found a solution to the problem; and (iii) a final *polishing* stage during which the solution was formalized and written. On each of these three episodes we report in two parts: (a) using a fragment from the group interview and (b) using corresponding fragments from individual interviews in which students reflect on the group-work that took place earlier.

4.1.1. Stage 1: Exploratory Stage (Group 6)—Group perspective

In this episode students read the problem, then interpret and discuss how to approach the required proof.

Group 6 [Anita, Bob, Donna, and Quin] is starting with the first problem: to show that the three perpendicular bisectors of sides of a triangle intersect at a single point.

Bob [1]: OK, let's start.

Donna [2]: Let's just fool around.

Quin [3]: [looks at the problem] So we have three perpendiculars and they meet at a point, and then knowing this

Donna [4]: Right

Quin [5]: ... and then knowing this we have to prove that ... no we have to prove they are concurrent.

Bob [6]: Yes. So we have to show ...

Donna [7]: We have three perpendicular ...

Bob [8]: We draw a perpendicular ...

Quin [9]: First thing we do ...

Donna [10]: I mean by proving we just kind of show that they intersect ...

Bob [11]: We got two perpendicular bisectors. Right?

Donna [12]: No, three.

Bob [13]: We want to draw here and show it's perpendicular. That's what we want to do. ... Right Quin? We got two perpendiculars. Of course they intersect. Right? OK. Now, this here, we draw first this perpendicular [he uses paper and pencil as he talks], then this perpendicular, and I gotta show this is a perpendicular bisector [pointing at the third line that he drew perpendicular to the third side]. I need to prove that.

Donna [14]: No we are not proving they are perpendiculars. We are showing they all intersect at one point. So we draw all three perpendiculars.

Notice that the group members are talking about two different things. Quin and Donna are interpreting the meaning of the problem that was posed [3–5, 7, 10], while Bob is thinking of the way to prove the statement [11,13]. At this point, Bob's thinking seems to be ahead of that of other group members. His idea is rejected by at least one group member [12,14]. We hypothesize that the rejection of Bob's idea reflects changes in meaning as his team mates internalize his partially externalized thoughts such as "I gotta show that this is a perpendicular bisector." [13] At this point, we can only note that there apparently are differences in thinking of group members; we cannot be certain of the differences of understandings or the reasons for those differences.

Later, Quin uses the GSP to draw a triangle and two perpendicular bisectors:

Quin [15]: Yeah.

Bob [16]: [continuing with his idea] Oh, maybe, show that it goes through that point [midpoint of the third side] and then show that's perpendicular.

Quin [17]: No, it's clearly perpendicular. We are given this.

Donna [18]: Yeah, we are given that [all three perpendiculars].

Anita [19]: [nodding, seems to be agreeing with Quin and Donna]

Bob [20]: No, no. You have to prove it by... that's perpendicular.

Quin [21]: You have two lines that meet at one point and the third also goes through that point but you don't have to show that's perpendicular.

Donna [22]: You don't.

Bob [23]: But what you gonna do is you gonna draw, I think, you have to draw this though that point. There is a line that goes through that point and then show that is a perpendicular bisector. That's the proof. Right Quin? [Bob is asking Quin for confirmation while Donna takes over and continues with construction using the GSP.]

For a moment all students focus on construction using the GSP. Donna constructed three perpendicular bisectors as lines perpendicular to the sides through the midpoints, and then constructed their point of intersection. When the construction was done, the discussion about how to proceed with the proof continues.

Donna [24]: They are saying . . . that's what we are expected to prove. The first is to show is that everything meets at G. The second proof is to show that G [pointing at the point of intersection on the screen] is the center of the circle. We have to prove that's, no matter what this triangle looks like,. . . you know [she takes the mouse, grabs one of the vertices and drags it around]. We are just to trying to prove they always meet at that point.

Bob [25]: First of all we have to prove. There are two things we need to prove. First one is that they are perpendicular [line bisectors]. That's a first thing. I think the other point is easier. You can show that they are all congruent, like they are all radii.

Donna [26]: I'm convinced but . . .

Quin [27]: We have to prove that they are. Basically, again . . .

Bob [28]: OK. [Again he draws, by hand, a triangle and two perpendicular bisectors].

Quin [29]: I remember you draw two lines and than you prove that a third line goes through the same point.

Donna [30]: Yeah. There is such proof but she does not want a formal poof.

Quin [31]: [Dr. —], In trying to prove we want to use proving by contradiction. What kind of proof are you looking for?

Researcher [32]: Any kind of proof that you as a group decide to be a proof. Anything that you negotiate to be a proof.

Bob [33]: OK. This is what I think we should do. You tell me if I'm wrong. OK. We draw this perpendicular. We draw that perpendicular. Right. Then you draw the line through this point. Now, it goes through this point. Now we gotta show that's perpendicular [and he draws another picture on his paper]

Donna [34]: All you need to do is to measure the angle. But this is informal.

Bob [35]: Yeah, but proving formally, all you need, all you gotta do is draw a third line [this time he seems to be suggesting to draw any line, not necessarily perpendicular to the third side as he suggested earlier]. You know these two intersect. We draw a third through this point. We show it is perpendicular and that these two are the same [pointing at two line segments on the third side of a triangle]. That's it. That's not very hard.

Donna [36]: But, see. We are not. . . [the researchers] is not looking for the formal proof.

Bob [37]: You mean informal proof? The GSP, measurements? [They all looked at the researcher standing near]

Researcher [38]: Whatever you as a group decide to be a proof. All I wanted differently from you is to write your proof in a paragraph form, not in a two-column form.

Donna [39]: OK. Then take off one of these lines that we constructed. [*Quin cuts a third line from their GSP drawing*.]

Bob [40]: Yeah. We want to show that's perpendicular. If we don't need to prove it formally, it's not that hard. I mean. . . [*Students have an extended discussion about how to use GSP to reconstruct the deleted third line*.]

Donna [41]: That seems to be a formal proof.

Bob [42]: We know much stuff. . .

Anita sat and observed this discussion quietly.

In this part, we see that Bob continues to develop his idea of the proof. His main strategy was to start with two perpendicular bisectors that intersect at a point and then to show that the third line passing through the intersection of the two other lines and the midpoint of the third side is perpendicular to the third side [13]. Quin and Donna clearly do not understand his suggestion [17,18], and point out that the third line is *given* to be perpendicular. Anita agrees [19] with Quin's and Donna's comments, leaving Bob alone with a different idea about the proof strategy than that held by the other three group members. Continually, Bob's externalized ideas were partially understood and therefore partially internalized by his team mates. This forced him to try to explain his ideas more clearly.

Bob tries to restate what needs to be proven [20]. Quin reiterates that the problem is to show that the third line is concurrent with the other two but not that it is perpendicular. Donna agrees with Quin [21,22]. Bob is able to express more clearly what he meant to say: If they construct the third line through the intersection point they would need to show that such a line is a perpendicular-line bisector of the third side [23].

His clarification was not useful to Donna. She says that the problem asks for two things: One is to prove that the lines are concurrent, and the second that the intersecting point is the center of the circle [24]. In contrast to Donna, Bob's clarification seems to trigger Quin's memory about the existence of such proof [29]. However, unable to recollect it, she turns to the researcher and asks if the group needs to prove the problem formally or informally [31]. Left with a choice to make—about the kind of proof—Bob continues with his idea to draw two perpendicular bisectors that intersect at a point, and then draw a third line through that point and show that it (a) is perpendicular and (b) cuts the third side in congruent segments [33,35].

Notice here that Bob suggests a slightly different idea from that given previously [13]. It appears he suggests that they construct *any* line through the intersecting point, then show that the line must be perpendicular to *and* must bisect the third side. Previously, he suggested constructing a third line perpendicular to the side and then showing that such line bisects the side. Although his idea regressed, it triggered Donna's thinking: Now she agrees with Bob's idea to start with two perpendicular bisectors [39] but not how to go about proving that the third perpendicular bisects the third side. It is important here to observe the strategy Donna suggested to get those two initial lines. She suggests they use the already constructed picture with all three perpendicular lines and remove one of them

[39]. This is a very important step in their problem-solving process that helped them later to construct their solution. At this point Bob consciously agrees to proceed with an informal proof. Was Bob's suggestion [33,35] just a slip of the tongue or a real regression of his previous idea [13]? Why did he agree to this compromise: Perhaps because the others accepted his idea to start with two perpendicular lines? Or was he tired of being different? Perhaps his own suggestion was not clear to him either?

These excerpts illustrate the complexity of students' interactions in a small-group setting. This complexity is the result of individual student's parallel functioning and the coordination of their internalization and externalization processes. Our observations reveal only how the group's strategy for the proof developed. Consistent with our theoretical framework, we can only speculate about how particular student's comments and actions represent externalization of their thought and internalized ideas that led to this developmental solution path.

4.1.2. Stage 1: Exploratory Stage (Group 6)—Individual perspectives

The students reviewed and commented on this segment during their individual interview. During the individual interview, they were asked to reflect and comment on what they remembered about the problem and the solution strategies that were developing during this phase. Their comments served to confirm some of our observations from the group session and also to provide insights that were not apparent from the initial videotape.

Donna [43]: We were talking and deciding what to do. It had to do what you construct and what you prove. He [Bob] went off one tangent and I went off for another. . . We were also negotiating on the definition of what you were looking for. Some of them wanted to say. . . that was another thing. We had to clarify that. Some were starting construction; it took a while for us to get together and agree. . . **Bob was thinking that the medians are the same as perpendicular bisectors**. That's why he kept going. I think he was putting the two definitions together thinking that connecting from the vertex to the midpoint was a median and it was perpendicular. [We assume that this was Donna's interpretation of Bob's explanation given in 16, 23, 33]. But see, we were able to talk such contradictions. We saw that wasn't true and we moved on. We sort of challenged one another. It was good to have others to explain. We all agreed about construction. At first, it was difficult.

Quin [44]: I just did not know how to approach this particular proof. So, then we thought we should use the GSP. I took the geometry course last semester and we had done this proof in class. Actually the professor did it for us and it was very long theorem proof. I remember thinking to myself – that's hard. And so I was thinking you want to reproduce the proof like that? And I had no idea how to start a proof like that – very, very complicated. ... We started with three bisectors as they were given and then we had to prove something else and that's why for the moment I was thinking – no it wasn't given, you'd have to prove that it was perpendicular bisectors. . . we did straighten each other out, you know one of us had information incorrectly. I think *Bob* was one of those. **He thought that the bisectors were not given and that you had to prove that it was perpendicular bisectors**.

Bob [45]: Well, we are just trying to get started here and we had a little trouble at the beginning. We needed to do the proof that these perpendicular bisectors of the triangle are concurrent and we were thinking about the ways to do that with GSP. We started off doing it just as we were doing a formal proof on paper and we always were making sure that everyone in the group had a good idea how to start the proof and what to do. . .

Anita [46]: We talked how to prove it. . . it takes me a while with proof. I just don't. . . I mean, I can sit there and look at something for ages. . . just looking something in geometry strictly it just does not make much sense. I did not say anything because I never knew what the sufficient proof is. . . I have no sense of logic. Analytical thinking, I'm great at. Like I can sit down and I can put things together or take things apart but proving things it's just something. . . just I'm not a logical thinker. And so at this point I was still trying to figure out what they were doing. . . I had no idea what they were talking about. . . Bob is a nice guy and I like to talk to him but he goes over my head. . .

These exploratory stage excerpts clearly show full verbal engagement of three group members as they negotiated their ideas with respect to the shared work. The fourth member was not verbally involved, but she was actively involved in making sense out of the conversation as we will see it in the next two stages of problem solving. The three members had two slightly different ideas how to approach the problem. Resolving this conflicting situation was an additional problem for the group.

Bob's initial idea was to construct two perpendicular bisectors, find the intersecting point, construct a third line through that point perpendicular to the third side and show it passes through the midpoint of the third side. Donna and Quin's idea was to construct all three perpendicular bisectors and somehow show that they are concurrent. At some point, Quin was trying to recall a formal proof of the problem that one of the professors had given in class. She only remembered that it was a very long and hard proof and worried about how to reproduce such a proof. Notice here that Bob and Donna seem to be working on the basis of *internal authority* (validity comes from their own deductive reasoning), while Quin seeks *external authority* (validity of ideas is given by a professor or textbook).

Donna and Quin's initial interpretations of Bob's suggestion are interesting [43,44]. Donna heard Bob's suggestion "you got to show that the third goes through that point" as a suggestion to construct a median and show that the median is a third perpendicular. She did the same after watching the clip during the individual interview, which indicates her inability to move away from her own thinking (*de-center*, in Piagetian terms) and recognize Bob's conflicting position. On the other hand, Quin's interpretation of the same suggestion from Bob was that "he thought that the bisectors were not given and that you had to prove that it was perpendicular bisectors."

It appears that Bob and his team mates did not share a common conception of their task, the proof strategy that they would pursue, during this exploratory stage. We also observed a fluidity of thinking by individuals during these interactions. As Bob struggled to clarify his proof strategy to the others, he switched between two ideas without seeming to be aware of the differences in the ideas he expressed. This probably made it even more difficult for his team mates to understand his suggestions. These changes in Bob's statements illustrate that *internalization* and *externalization* are not inverse, functional operations. As Bob transforms the representations of his ideas, the ideas themselves change. It does not seem that this is simply his attempt to clarify the ideas for the others, but that it also reflects a simultaneous refinement of his thoughts about the proof. It is exactly this refinement of thoughts that indicates cognitive change for an individual.

There is an interesting pattern of conversation by this group: One partner begins a sentence or an idea and the other partner completes it. To us, this suggests that the students are accustomed to working closely with each other's ideas. It seems likely that the individual *zones of proximal development* of students in this group often overlapped because of the way they could complete each other's thoughts. This allowed them to move beyond initial miss-communication to achieve success with the problem. For example, Bob said in his individual interview "I made a mistake and said that you got to show it's perpendicular but what I meant to say was that it's a midpoint. She [Donna] corrected me—she knew what I was thinking but I was saying the wrong thing." As we will see, that success did not always mean that they resolved all of the misunderstandings that were apparent at this stage.

4.1.3. Stage 2: Productive stage (Group 6)—Group perspective

In this stage students are working on the problem solution, trying to find or create the proof. After the group agreed upon the problem (agreed in the sense that no group member had any question or comment) they proceeded with their proof. First, they constructed an arbitrary triangle and three perpendicular bisectors. Next, they constructed a point of intersection of the perpendicular bisectors. They deleted one of the lines and then disagreed about how to replace the perpendicular bisector. Bob suggested drawing a third line through the intersecting point and the midpoint of the third side, then showing that it is perpendicular to the side [note that this is his new idea]. Donna suggested drawing a line through the intersecting point that is perpendicular to the third side, then showing that it bisects the third side [Bob's first idea that he had abandoned at this point]. This excerpt, which is the continuation of the excerpt that ends at [14], illustrates how the disagreement was resolved.

Bob [47]: Select the midpoint of the segment.
Donna [48]: No, we just want to say construct a perpendicular line.
Quin [49]: No, we don't know it's perpendicular.
Donna [50]: No, we gonna . . .
Bob [51]: You could draw a perpendicular through H and. . .
Donna [52]: **We are doing it backwards**. [they used 'undo' and 'redo' feature of the GSP as they discussed]
Bob [53]: You can draw a perpendicular through H and. . .
Donna [54]: Draw a perpendicular. . .
Bob [55]: You could do that . . .
Donna [56]: Yeah, end then show that's . . .
Bob [57]: . . . perpendicular, I mean. . .
Donna [58]: . . . GSP measures to see . . .
Bob [59]: . . . show that these two are same. . .
Quin [60]: I think you have to show that is perpendicular and bisecting also . . .
Bob [61]: Ah. . . no what you need. . .
Quin [62]: The problem asks us . . .you have to prove it's perpendicular, also . . .
Bob [63]: I think you draw it through the midpoint of that segment.

Quin [64]: We don't know they are . . .

Bob [65]: When it goes through the midpoint that it is perpendicular. . .

Anita [66]: . . . perpendicular bisector. . . you don't know

Bob [67]: You gotta draw through the midpoint of AC and then show that's perpendicular. Right?

Quin [68]: No, I think you have to show it's perpendicular and bisector, both.

Bob [69]: You can't do that because you can do the line through the midpoint anywhere on that. . . I don't know. . . you don't know. . . you draw through the midpoint of AC and then show it's perpendicular.

Donna [70]: Yeah . . .

Bob [71]: But you can't do it any other way. You can draw a line through H. . . and then show it's perpendicular . . . Doesn't necessarily mean . . .

Quin [72]: . . . actually you did not construct a midpoint. You constructed a perpendicular but not a midpoint.

Bob [73]: How . . . Oh, I thought you constructed a bisector through a midpoint.

Donna [74]: You can draw a perpendicular line and then show it's the midpoint.

Bob [75]: Oh, I know, I know, **I made mistake**. . . You've got to draw a perpendicular and then show it's a midpoint. You can always draw a perpendicular but you can't say it necessarily goes through a midpoint. If you did that and . . .

Donna [76]: I think

Anita [77]: Construct a perpendicular

Donna [78]: OK, now highlight that point of intersection. You need to show . . .

Anita [79]: Highlight that point. [Quin highlights the point and constructs a line through it perpendicular to the side]

Donna [80]: You need to show it's a midpoint. **Use your measure now**. [Important note]

Quin [81]: We need to construct . . .

Anita [82]: Measure distance . . .

Bob [83]: Go to measure . . .

Anita [84]: Highlight the point [all of them are talking at the same time]

Donna [85]: We know that's perpendicular. We need to show that goes through a midpoint.

Bob [86]: Yeah.

The group continues with construction and writing of their proof. Their complete solution is presented later, in Fig. 2.

In this episode we see the group struggle to prove that the third line, which is concurrent with the other two perpendicular bisectors, is a perpendicular bisector of the third side. Bob's new idea—draw a line through the intersecting point and a midpoint of the third side and then show that it is perpendicular [47]—was not clear to other group members. Donna suggested starting by constructing a perpendicular line through the intersecting point, an idea originally suggested by Bob—did she realize this? [13]. It appears that her idea came as a result of removing the third line [39] and "doing it backwards" [52]

Given triangle ABC, construct a perpendicular bisector of AB and BC. Mark point of intersecting bisectors as H. Construct a perpendicular from H to segment AC. Label point of intersection between perpendicular and AC as L. Measure lengths AL and LC to show they are of equal measure. By dragging different vertices to form different arbitrary triangles, segment AC continues to be bisected by perpendicular HL; therefore, the perpendicular bisectors of a triangle are concurrent at a single point. Construct a circle choosing H as the center and any one vertex. Measure the radius of the circle, and lengths HC, HB, HA. All lengths are equal, which confirms that all vertices lie on the circle. Therefore, the concurrent point H is the center of the circle that circumscribes the triangle ABC.

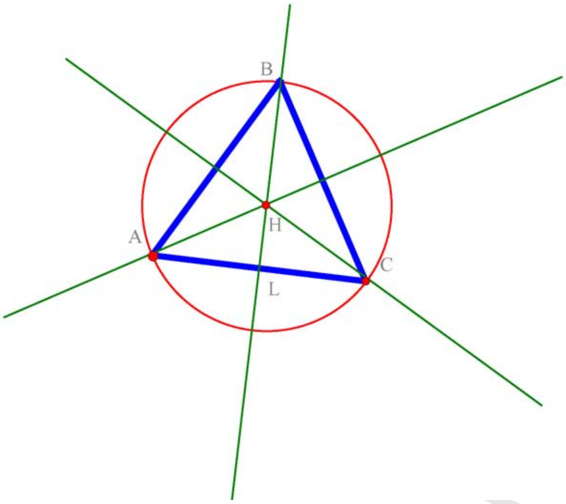


Fig. 2. Group 6's written solution.

using the GSP. At first, Quin seems to be confused with both ideas. She focused on the statement of the problem, insisting that they need to show that the third line is perpendicular and bisects the line segment [68]. Her suggestion was not clear to other group members and may not have been clear to herself, either. Her attention clearly was focused (*centered*, in Piagetian terms) on the fact that the third line has to be a perpendicular bisector. Bob sees the difference between their suggestion and his own idea and acknowledges his "mistake," which actually was not a mistake [69,75] but a valid alternate solution strategy. The group eventually agrees on the solution that uses this 'new' idea of Quin and Donna, which they seemed to think was what Bob suggested.

We want to emphasize here that the GSP played a very important role in the process of de-centration of group members' thoughts and made it possible for some group members to move forward in their thinking. After some discussion about the problem, as given above, this group used GSP to construct a triangle and its three perpendicular bisectors. They manipulated vertices and noticed that the bisectors remained concurrent. Then they decided to cut one of the perpendicular bisectors from the GSP figure. A discussion ensued about how to reconstruct the deleted line. We see here that GSP is being used as a manipulative by the group. Following this construction and manipulation, the thinking of several group members apparently took new directions. We emphasize that the point here is not the *mathematical correctness* of their argument, but the *ways in which the students learned from each other and as a collective*. For example:

1. It is clear that Donna's thinking about the solution of the problem was influenced by constructing the third perpendicular bisector in GSP, removing it, then discussing how to put it back in GSP. [GSP provides two methods for constructing this line: (a) construct a line perpendicular to a side through a given point, and (b) construct a line through the midpoint of the third side and the point of intersection.] Describing how the software helped her move beyond initial confusion, Donna said "But we saw that wasn't true and we moved on. . . I think that's what is good about group. It was good to have others to explain. We all agreed about [the GSP] construction. At first, it was so difficult."
2. Bob abandoned his idea about the midpoint once he understood Donna's suggestion and after seeing the construction (using a perpendicular to the third side).
3. Quin was able to see that in order to show that the third line is a perpendicular line bisector, one has to assume one property of the line and show the other (for example, assume the line is perpendicular and then show it bisects the line segment).

It is important to note that so far the group had followed a deductive line of proof. Excerpt [80] illustrates the moment when the group switched from deductive to inductive proof: "Use your measure now." Significantly, this quote suggests that the measurement feature of the GSP software provided an opportunity for the group to show congruence of the two segments informally, and thus to turn from deductive proof to inductive verification. We wonder how typical this type of action might be in the dynamic geometry environment. Certainly, it indicates the heightened importance of discussing what constitutes a mathematical proof in the inductive, dynamic-geometry environment.

This episode clearly illustrates how individual internalization and externalization moves can lead to changes both in individual and group conceptions of a problem with the final group state not completely attributable to any individual's conception. We observed a group negotiate and construct shared meanings when the group members listened to each other, considered each other's perspectives, and even abandoned their own ideas when it seemed necessary. The teacher's role in helping students to achieve these objectives—such as pointing out some significant comment of a group member and asking others to think about it—is crucial in small-group settings. Depending on the task and pedagogical objectives, this guidance could be provided either during small-group sessions or during subsequent whole-class discussions.

4.1.4. Stage 2: productive stage (Group 6)—Individual perspective

Students again reflected on the group work in a subsequent individual interview.

Donna [87]: OK. Bob is saying it wouldn't be that hard to prove it formally in two column proof and we all agreed. At this point we all agreed that this is how to construct. Once you show that the two segments have to be the same then you are done. He [Bob] thinks that if we really go to the formal definition, it wouldn't be that hard. We sort of taught him how to use the GSP. . . . We went and we drew two perpendicular bisectors. How did we really do? Look at this shape. We marked these. We marked these two bisectors. OK. We drew two perpendicular bisectors, we did a point of intersection, we drew like a perpendicular to a third side passing through that intersection. So, we are not proven that and we gonna prove that this is a perpendicular bisector that bisects a line. . . . The GSP constructs it perpendicular and we are . . . so if these two parts are equal then in fact it is a perpendicular bisector. And that's what we did; that's what we measured here, these two segments stretching them around to show that a third line going through H is indeed a perpendicular bisector.

Asked to reflect on what Bob said [63] Donna added:

499 *Donna* [88]: B was. . . If you do this and. . . We did this on our screen to show that it is a bisector. If you go to the midpoint and show that it's 90, it's another way. I think that the fact that
500 the three girls, that we agreed that it was that way and then we went to do it and Bob wanted the other way. There were the three of us who pull it one way.

501 Quin and Bob expressed similar views of the episode, suggesting that they ultimately shared a conception of the work they had completed:

502 *Quin* [89]: What happens here is that I'm suggesting to the other members of the group that maybe we should prove that theorem by assuming that two of the lines, two of three lines intersect at one point already and then show that the line going through that point has to be a perpendicular bisector, meaning that it has to bisect the other side and form right angle at it. . . . Well, we had the two lines that intersect at the point. We, my suggestion was to draw an arbitrary line through that point. We don't . . . not a perpendicular line, just an arbitrary line and then prove it that that line is perpendicular and bisecting the other side. . . . Well, I shouldn't say that. I guess at the time I just thought there was just one line going through it that would be the exact line and it was supposed to be perpendicular. I guess you have to assume that it is perpendicular and prove that it is bisecting the side. . . . I guess my suggestion at that time was . . . I didn't want to think too much. I wanted to prove without assuming what I'm trying to prove. But I thought that assuming that it's perpendicular you are assuming what you are trying to prove. . . . I think we ended up drawing an inclined
503 line through the point of intersection and then proving that it was bisecting the segment. . . . We measured the segments and showed that the two parts are equal.

Bob [90]: Yeah, what happened was. . . I was drawing it right but I was saying it wrong. So, I was saying that when you drop that perpendicular to the side it goes through the point of concurrency. I made a mistake and said that you've got to show it's perpendicular but what I meant to say was that it's a midpoint. She [Donna] corrected me—she knew what I was thinking but I was saying the wrong thing. . . . My idea was that you construct two perpendicular bisectors to two sides of the triangle and then you must drop a perpendicular through that point of intersection to the other side and then my idea was to prove that went through the midpoint of that side. That's how you would do it formally. I don't know if that's right. I think they were doing little different. They constructed three and then I think showing that is the point of intersection of all three or something like that.

504 The interviewer commented to Bob that Donna seemed to agree partially with his idea. Bob responded
505 in a way that shows there was not complete agreement or understanding of each of their views about the solution strategy:

506 *Bob* [91]: Yeah, it was really incredible what she was talking about the whole time. I still even at this point don't understand totally what she was trying to suggest. I felt like she had ideas about
507 and stuff what to do. I was unclear the whole time while we were working about what she was talking about. I think she had some idea that I was not familiar with or something and I was trying to piece through what she was saying and I'm still not totally what she was suggesting but. . . I mean, originally, I had the idea to do what we just did.

Anita was quiet during the whole discussion on how to approach and solve the problem. When the interviewer asked if she understood what the group was discussing and why they discussed using a GSP proof or a formal proof, what it was that bothered them, Anita said:

Anita [92]: I don't think so. I think . . . I was trying just get it done. I always like . . . I guess to me if I do something on my GSP or something, to me the proof is just write down what you did. It doesn't matter what theorem you are formulating or that. I thought they were coming getting caught up and they kept talking about formal proof and I was sitting back there thinking – you just did it, that is a formal proof in itself. But I've just set there. . . . With the group you know I'm there . . . I'm usually one that does the writing. I don't do much speaking unless I knew what we are doing. Like one-on-one it usually takes me easier. When there is more than three people I'm very easily intimidated.

From these individual students' reflections we see that Donna did not understand that Bob's latest suggestion was not wrong and that she was very happy with the group's final solution [87]. Her perception was that Bob insisted on a formal proof. One wonders if that might have blocked her reflections on Bob's suggestion. Only later, after Donna was asked to reflect on Bob's suggestion during the individual interview, did she realize that his suggestion was valid [88].

The group's influence on Bob's thinking is quite intriguing. Even during the individual interview he was convinced that his initial idea was wrong [90]. The interviewer noticed this critical incident in which Bob appeared to change his thinking because of group influence. After hearing and understanding Donna's suggestion, Bob changed his mind. We can only speculate that Bob rejected his initial idea believing that there is only one solution to a problem. During the individual interview, this portion of the video was replayed several times as the interviewer probed Bob about how and why his thinking changed. He explained that he was actually thinking like Donna, but "I was thinking one and saying another thing." Although he seemed to remain convinced that his original approach was wrong, it is interesting that during the individual review, while watching the same clip, Donna came to believe that either approach would work. This illustrates Valsiner's notion of the co-construction of new ideas by individuals when solving a problem collaboratively. This also illustrates Bob's ability to *decenter* (Piaget's notion) from his own thinking and consider the perspective of others.

Quin explained her misconception and emphasized her contribution to the group's solution [89]. Anita's explanation indicates that she did not comprehend the group's discussion and the issues that group members raised while working on the problem [92]. It was only after completing the GSP construction that she showed some signs of understanding of the group's solution, while still acknowledging her inability to distinguish between the formal and informal proof.

Bob's individual comments illustrate the complexity of the interactions that take place in this sort of mathematical problem-solving activity. He admits that he did not really understand the ideas that Donna expressed. Still, Donna's comments shaped Bob's thinking about the problem, and the group was ultimately able to agree that they had achieved a solution to the problem. This highlights two important characteristics of the theoretical framework: (a) Bounded indeterminacy holds that we should not expect a clear, causal relationship between stimuli and responses, and (b) an individual's *zone of proximal development* describes a readiness to make use of external stimuli to refine internal conceptions. Group members were influenced by the discussions that took place, but this influence was not simply a matter of adopting the ideas expressed by others. The ideas they heard triggered (*or, in Valsiner's terms, canalized*) advancements or, at least, changes in their thinking, but these changes had a unique and personal character that was not exactly shared by all in the group.

4.2. Stage 3: Polishing stage (Group 6)—Group perspective

In this stage the same students are writing their solution, having substantially developed their ideas about the problem.

Anita [93]: Given a triangle ABC,
Quin [94]: no
Bob [95]: draw a perpendicular BL and then the other one . . .
Quin [96]: No BL,
Anita [97]: It doesn't matter which one . . .
Bob [98]: Yeah
Anita [99]: Construct a perpendicular bisectors AB and BC. Any two. Mark the intersection H . . .
Bob [100]: Mark the point of intersection H . . .
Anita [101]: and then construct
Bob [102]: Now draw a perpendicular from H to L
Quin [103]: Yeah, that's what we were doing
Anita [104]: through a midpoint
Bob [105]: From H to a midpoint of the segment.
Quin [106]: We already know that L is a midpoint.
Bob [107]: Draw a perpendicular from H to AC at point
Donna [108]: . . . through that H and then draw a perpendicular. . .
Anita [109]: Now measure AL,. . .
Donna [110]: Wait a minute.

Anita continues to dictate: "Construct a perpendicular from H to segment AC. Label the point of intersection between perpendicular and AC as L. Measure lengths AL and LC to show they are of equal measure."

It is evident in this excerpt that some group members were still confused with two different ideas [101,104–106]. But together they were able to reconstruct their previous discussion and write the solution (Fig. 2). As they finish writing the solution Donna takes over typing while Anita dictates the rest of their description.

In this episode the fourth group member, Anita, became active. She felt confident that she could describe the construction and could repeat the group's "GSP proof," so she stepped forward to make her group contribution. This excerpt again illustrates the pattern of conversation, so typical of this group, with individuals completing sentences of other group members.

This group's written solution for the problem is presented in Fig. 2.

4.2.1. Stage 3: Polishing stage (Group 6)—Individual perspective

During the individual interviews, all students stated that by the polishing stage of their work on the problem the group had reached agreement. All became active and participated in wording and writing a solution. Additionally, they all said that Anita became more active at this phase and was the one who dictated most of the written solution.

Anita's role in this final stage provides another illustration of how students, when their zones of proximal development appear to overlap, are able to refine their understandings based on the ideas expressed by others. These refinements are not a deterministic, uncritical response to external stimuli but involve choices between conflicting points of view. Anita had not been involved in the previous discussions, but clearly had been following and came to make her own sense of the ideas that had been discussed. She was not simply recalling previous ideas of one or more group members, but had internalized them and was able to produce her own externalization, which differed from ideas expressed by other group members during this stage.

For example, in the first few lines of this final excerpt, Anita responds to the disagreement of Bob and Quin by saying that the choice of which two perpendicular bisectors they start with does not matter. She also has the idea of constructing a line to the midpoint of the third side and *showing* that it is perpendicular to the side, rather than *constructing* a perpendicular to the third side as Bob suggests during this exchange. She does not seem confused by these contradictory suggestions or by differences in ideas from Bob, Quin, and Donna. Instead, she seems to have her own understanding that she relates selectively to the comments made by other group members. Anita refines her ideas through selective adoption of ideas expressed by others, not from an uncritical adoption of every idea that comes up. For this reason, we believe that much of the group discussion took place within Anita's zone of proximal development. She was able to internalize then externalize the ideas, even though she had not participated in the discussions previously, because of her readiness to interact with the ideas.

4.3. A brief overview of the remaining group sessions

We have presented a detailed analysis of one group's work on a single problem. Group 6 was chosen because their exchanges help illuminate the mechanisms of co-construction. Several other groups found proofs for one or more problems. Each of these successful groups had similarly rich interactions as students refined and melded individual ideas to produce a shared, group understanding of the task. The nature of interactions of Groups 2 and 5 were very similar to those of Group 6, although Groups 2 and 5—in contrast to the decision of Group 6 to give an informal verification based on a GSP construction—gave formal proofs and did not use the GSP except to produce figures to accompany their proofs. Their group sessions involved an extensive exchange of ideas, balanced contributions from group members, and fairly high level of comfort in completing each other's thoughts. Group 5 never wrote out a formal proof, but were satisfied that they understood completely what was necessary to prove the result. From the videotape of their group session, we know that their impression was accurate in this respect.

Each group's shared or accepted understanding of the nature of the task strongly shaped the interactions that followed. For example, the decisions by Groups 1–5 to pursue a formal proof rather than informal verification relegated the GSP to an insignificant role. Only two of these groups even used the software. One group used the GSP as a graphics program to sketch figures, the other only to confirm at the very end of the problem-solving session that their conjecture that the inscribed figure in Problem 2 was a parallelogram.

The significant influence played by the nature of the group's shared vision of the task, or lack of a shared vision, is also illustrated by the work of a group that was much less successful. We observed a markedly contrasting style of work by Group 1, which was unable to give a solution for any of the problems. This group essentially worked in isolation, with two individuals (Kerry and Will) seeming to wait for the third member (Nathan) to come up with an idea or solution. The group worked in total silence for periods of 10–15 min, broken by brief exchanges in which group members asked whether the others had made any progress. They could not seem to come up with the proof ideas they wanted and seemed unable to generate and refine speculative ideas to move forward. Their most animated exchange took place near the end of the interview when they were asked to summarize their progress before the end of the session.

From their transcript, it was apparent that this group could have given an inductive "GSP solution" of the second problem had they wished; and one would suspect they also could have managed that for Problem 1. Their understandings of the capabilities of GSP were much more closely shared than their views of the nature of proof, so the exchanges when using the GSP seem more productive than at other times (more like those of Group 6, described above). Nathan, the leader, was intent on giving a formal, axiomatic proof. Kerry mentioned a wish to refer to a book to find some axioms or theorems that could be used. Her view of proof seemed to be that it involved recalling arguments and statements made by others, perhaps by the instructor or in the text. These two students had very divergent views of mathematics and, specifically, the nature of proof: Nathan looked *internally* for meaning and authority, while Kerry depended entirely on *external* sources of validity and verification. Despite their different understandings of the nature of proof, both sought a formal proof. That is in contrast to Group 6, above, which sought an informal, inductive verification using the GSP.

It seems that several characteristics of some groups allowed them to collaborate more effectively in this problem-solving activity. First, the group members needed to share an understanding of what was involved in providing a mathematical proof. It was not important that this shared understanding be of a formal proof, but that the students agreed, at least implicitly, about what they had to do, even if that was informal, inductive verification of the statements. Second, the group members had to have understandings of the problem situation that were similar enough to that of group members that they could interact with the ideas of group members. This required an ability to internalize externalizations of other group members and to relate their own ideas to those of other group members. As we saw with Group 6, it was not necessary that understandings were shared by sender and receiver: That group had several instances where one group member appeared to misinterpret the ideas expressed by another. What seems important was that the ideas offered by one person were within a zone of proximal development of another group member. That is, the understandings were close enough that the person could interpret the new ideas within the context of their existing understandings.

Several groups spent most of the time working unsuccessfully as individuals in silence, with brief conversations to see whether the others had made any progress. For example, the members of Group 1 appeared to be extremely reluctant to reveal their thinking before they were certain that their thinking was correct. Members of this group, including two students who were quite successful in the geometry course, claimed during their individual follow-up interviews that their group work was not representative of the style of work they mostly had used during the term. Other groups showed a similar inclination to work privately and individually during parts of the session.

In summary, then, we observed widely varying degrees of collaboration and interaction, and these seemed related to the success that groups had with the problems: Groups with more extensive and open discussions seemed to have greater success with the problems. A group whose members held differing understandings of proof, such as members of Group 1, did not seem able to have productive discussions about their work on the problems. We believe that this lack of a shared purpose or common understanding of the task may have blocked their ability to function productively as a small group.

5. Findings and implications

5.1. Group influences on individual thinking

We were able to observe several instances in which groups influenced individual thinking. This often moved the individual and group closer to a solution, but we also saw one case in which a student (Bob in Group 6) abandoned correct reasoning to adopt an alternate—and also correct—strategy.

The group interactions we observed in productive groups, especially Groups 2, 5 and 6, matched the behaviors we would expect based on the co-constructivist framework. Excerpts from the work of Group 6 reveal students expressing ideas of their own and of others in the group. As they internalize, then externalize the ideas, their understanding of the problem changes. Their new expressions of ideas appear to influence the ideas of team mates with the cyclic process moving the group's conception of the problem toward a solution acceptable to the group and not reflecting the isolated work of any individual.

Students in this study were aware of the contributions made by team members and seemed to recognize some of the small-group processes that contributed to enhanced, deeper understanding of mathematics. One might expect to discover that individuals in a group held different views of their own and their group members' contributions to the tasks. We were somewhat surprised to discover the extent to which group members shared common views about themselves and their team members. This includes what they did during the problem-solving session and during a prior geometry course. This agreement and the generally favorable views of the efficacy of group work suggest that cooperative and collaborative group activities helped students better understand the thinking of their classmates—and, consequently, their own.

The work of several groups seemed to show how productive interactions advanced the group's efforts to find a proof by influencing individual group members' understanding of the problem. In contrast, another group that had a dominant leader (Nathan, Group 1) who was expected to solve most problems had little interaction because he was unable to discover a productive approach to the problem. As group members reported, they usually reacted to his ideas but rarely generated their own. Although one of his group's members—Kerry—had a limited notion of proof, there was no evidence that her thinking was influenced by the comments of Nathan. We believe that Nathan was not making suggestions that were in Kerry's *zone of proximal development*, so the insights of Nathan could not influence the thinking of Kerry.

Several participants expressed highly favorable impressions of the impact that small-group work had on their learning of mathematics during these courses that involved extensive cooperative learning. This was in spite of the common view that it involved more time and effort. Their comments also illustrate their own recognition of the contribution of interactions with others to their own understanding of mathematics. For example, Kelli (Group 2) responded to the question of how the group work had influenced her learning by stating

I think I remember everything a lot more because you not only have to prove it to yourself but you have to defend it. It makes you have to go over it at least twice. I'll remember it a lot more.

Kelli is expressing the idea that it was not enough to internalize an idea in this course, but that she had to repeatedly externalize the ideas to share them with group members. She believed that these repeated transformations from thought to expression and back helped to produce better understanding of the material. Similar sentiments were echoed by others. For example, Al (Group 2) stated:

[Group work] really helped. At times I would have been lost without guidance. . . I got a lot out of this course. Sometimes [in other courses] you just go through and do the homework problems and take the tests. But I feel I have a better understanding and can recall it. The groups were more fun and we didn't get bored with it. It was a lot of work but worth it. Sometimes you can skate your way [through a course] but with people helping you they can point out where you went wrong.

6. Concluding remarks

Students need to discuss mathematical ideas to develop rich and deep understandings of important concepts. This view is widely shared by educators involved in the teaching and learning of mathematics at all levels, from elementary school through graduate programs in mathematics. Many have embraced a constructivist view of learning that has been developed in various forms from the early psychological and sociological work of scientists such as Piaget and Vygotsky. Valsiner more recently has described *co-constructivism*, a theoretical perspective that blends an individual, psychological perspective on intellectual development with a group, sociological perspective. This theoretical perspective holds that development of knowledge in a social setting is bidirectional. Individuals process ideas through interplay of internalization and externalization processes, and are not only influenced by the social culture of knowledge, but also have an influence on the social culture. It is these transformations of ideas between group and individual and between external and internal representations that allow the group and individual to co-construct mathematical ideas in ways they find meaningful.

This study supports the view that small-group work can have significant, positive effects on student learning, problem-solving, and self-confidence in mathematics. The co-constructive view of learning is consistent with interactions that we observed in many of the groups, especially the three groups that had productive discussions leading to solutions of one or more problems. Given the *bounded indeterminancy* that underlies the theory, we did not expect that every group would exhibit the interactions described by the theory. Indeed, Vygotsky's notion of readiness—the individual's zone of proximal development—suggests that there are necessary conditions for such social interactions; bounded indeterminancy says that these conditions are not sufficient to ensure such interactions will, in fact, occur.

It is clear that many factors contribute to the nature of interactions that take place in small-group settings and their impact on individual thinking. For example, students must be operating within the zones of proximal development of their group members if the ideas are to be modified and developed through group interactions. Group members must share an understanding of the nature of their task, such as having a common notion of proof. It is possible for groups to work *cooperatively*, in the sense of division of labor, without also having productive *collaborative* interaction, in the sense of working together to achieve a common goal as a team.

The research methodology employed here could be used in a variety of settings to generate useful data about the ways that students learn mathematics in a social context. The use of group sessions with minimal intervention, followed by individual probing based on video recordings of the previous session allows the researcher to observe activities without directly influencing their course, but then probe individuals in some depth about their thinking during the group session. Students' reflections and interpretations of particular group sessions could give more insights than any interpretation that researchers can get by just observing the video tape. For example, individual interviews with members from Group 6 revealed that *difficulties the group experienced in their problem solving session were a result of poor attention the students were giving to each other's comments and suggestions*. Reviewing a clip from their group work during the individual interview, Donna realized that Bob's suggestion how to prove that the third line is perpendicular to the side was also correct. This is a *significant result documenting the importance of students' reflection and could be used as an instructional strategy*. The video of the previous session provides a good stimulus to generate individual reflection by the student. Technologies also provide the opportunity to study a dynamic record of students' written responses to better track the development of ideas over time. Further studies conducted in a variety of settings could provide insights that could guide instructors in the sorts of group activities and modes of instruction that would facilitate the learning of undergraduate mathematics by providing data that helps to further refine and develop the theories of learning in a social context. Such studies may also provide better insights to the characteristics of small groups that seem to contribute to particularly rich and effective interactions and collaborations between students.

We believe our research, as described in this paper, sets a foundation for further investigation of the use and impact of small-group activities in the teaching of mathematics. The most significant ideas we discussed include:

- The learning of mathematics can be enhanced by promoting the development of shared knowledge and the individual learning within a context of small-group work.
- Valsiner's notion of co-constructivism provides a theoretical framework that helps to explain the construction of knowledge by individuals who participate in mathematical activities with others. This theoretical perspective holds that successive changes of representations of ideas, through the processes of internalization and externalization, promote changes in and refinements of both individual and shared mathematical notions. Our study suggests that instructional emphasis on reflective thinking about externalised thoughts of others in the group is a very important aspect of parallel functioning.
- Through observation of mathematical activities in a small-group context and subsequent stimulated individual introspections on the group activities, we were able to explore the individual and social dimensions of mathematical learning. This methodology seems promising for researchers investigating the learning process in a social context because of its focus on both individual and shared notions, and seems applicable not only to other mathematical domains but also to other disciplines.
- Individual cognition, a psychological perspective, and social construction of knowledge as described by Vygotsky are interactive dimensions of learning. For example, we observed that productive social interaction seemed to require individual understandings that could be shared, to some degree at least, by group members: The groups had to share an agreed understanding of the requirements of the task, such as a notion of acceptable proof. This is in keeping with Vygotsky's notion of the *zone of proximal development*.
- Teachers should be aware that while a technology-enriched environment can be beneficial to development of students' understanding of mathematics, it may require specific instruction about expectations for a given task. For example, in our study we saw that Group 6 constructed their proof using 'undo' and 'redo' (or, cut and paste) features of the GSP. By undoing and redoing the third perpendicular,

students were able to move forward. At the same time, using the GSP's measurement feature, students switched from their formal mathematical proof to an informal and inductive proof. We believe that clear communication of our expectation for a formal, mathematical proof would have changed the nature of proof that this group produced.

Uncited references

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