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Rationals and decimals as required in the school curriculum
Part 1: Rationals as measurement

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Abstract

In the late seventies, Guy Brousseau set himself the goal of verifying experimentally a theory he had been building up for a number of years. The theory, consistent with what was later named (non-radical) constructivism, was that children, in suitable carefully arranged circumstances, can build their own knowledge of mathematics. The experiment, carried out jointly with his wife, Nadine, in her classroom at the École Jules Michelet, was to teach all of the material on rational and decimal numbers required by the national program with a carefully structured, tightly woven and interdependent sequence of "situations." This article describes and discusses the first portion of that experiment.

Keywords: Rational numbers; 4th grade; *Didactique*; Theory of situations; Didactical engineering

1. Introduction

1.1. Warfield

Among the most attractive and accessible features of *Didactique* are the examples of specific situations. A number of particularly well-known examples come from Nadine and Guy Brousseau's "Rationals and Decimals in the Required Curriculum." I was particularly pleased, therefore, when in the early stages of my introduction to *Didactique* I was given a copy of the book. At the time I saw it as an attractive and coherent sequence of lessons about fractions and decimal numbers and as evidence of the degree to which the field of *Didactique* is based in the reality of the classroom. As I delved deeper into the book, I discovered that it also contained thought-provoking commentaries on the learning to be expected from specific situations (as well as some learning that should not yet be expected), and a pair of articles discussing the enormous mathematical and pedagogical study behind the choice of the sequence.

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That was in 1992. In the ensuing decade I have read more *Didactique* and have had more conversations, and repeatedly have had the experience of discovering that there was yet more to the book than I had realized. I was therefore delighted when Guy Brousseau mentioned his interest in returning to the book and writing down some of the additional thoughts and discoveries he has come up with since the original writing in 1987. We settled on making it our next joint project.

As we launched the project, some things immediately became clear. One was that simply taking the original book and adding a decade and a half's worth of further thoughts would produce an object more appealing to a weight-lifter than to a reader. We therefore decided that only a very few lessons would be reproduced in their entirety, and the rest would be condensed. Even at that, though, the project remained unmanageably bulky and not conducive to the kind of deepening of understanding that Brousseau had in mind. So we broke off the first three modules, which deal with rational numbers as measures, and postponed the rest. Brousseau's introduction, which follows, provides enough of a description of the rest of the book to put this part in context.

There then remained the issue of putting all this in a form that would be of use to the reader. One clear element was to insert commentaries between the lessons, condensed or otherwise, where they would be pertinent. Even that supplied a challenge, because a concise commentary requires a precision of vocabulary which needs to be understood as such. Details on that front appear in the first of the commentaries.

One other need must be addressed, and that is one which pertains to the English-language readership, and hence must be addressed in this part of the introduction. Our language, although generally rich in synonyms, fails to supply us with a pair of words to correspond to the French near-synonyms "*savoir*" and "*connaître*." Both are translated as "to know." Likewise the nouns associated with them, "*les savoirs*" and "*les connaissances*" are generally both translated as "knowledge." At times this is fine, at other times it is a problem. The latter is notably the case in dealing with *Didactique*, where the words are frequently and highly intentionally distinguished. In the following paragraphs we will attempt to make the distinction clear, and we will propose a solution to be used in this article.

At a first pass, the verbs can be thought of as "to be familiar with" (*connaître*) and "to know for a fact" (*savoir*). For some examples the distinction is clear and useful: "*connaître*" a theorem means to have bumped into it sufficiently often to have an idea of its context and uses and of more or less how it is stated; "*savoir*" a theorem means to know its statement precisely, how to apply it, and probably also its proof. On the other hand, when it comes to an entire theory, with a collection of theorems and motivations and connections, what is required is to *connaître* it. *Savoir* at that level is not an available option — but on the other hand, no real *connaissance* is possible without the *savoir* of some, in fact of many, of the theory's constituent parts.

The corresponding distinction exists between the two words for "knowledge," with the additional complication that each of the French words has both a singular and a plural form.

Before offering a solution, I propose to give an example of a way in which having the two words is both thought-provoking and a material aid in analyzing what's going on. Currently in American mathematics education there is considerable debate about the status of certain kinds of knowledge. One side is accused of interesting itself solely in "skill-drill" and computation, the other of interesting itself solely in "fuzzy math," where anything goes as long as it is in the right general vicinity. Consider instead the following description: all school learning is an alternation of *savoirs* and *connaissances*. Isolated parts are acquired as *savoirs* connected by *connaissances*. Without the *connaissances*, the *savoirs* have no context and are swiftly mixed or lost. Without the *savoirs, the connaissances* are more touristic than useful. Imbedded in *connaissances*, *savoirs* can develop gradually into a solidly connected chunk — in fact, a *savoir*, which is then available to be set into a wider *connaissance*. Thinking this way then provides a tool for contemplating another of the current hazards of mathematics education: assessment. It is a clear need, but a thorny issue. And one of the causes of its thorniness is that all that can be assessed on a standardized test is *savoirs*. The state of a student's *connaissances* is visible to the teacher if enough time in the classroom can be devoted to the kind of activity where *connaissances* are built and used. But an over-emphasis on visible, "testable" knowledge leads to attempting to teach the *savoirs* without the *connaissances* to hold them together and carries with it the danger of damaging the entire fabric of the learning.

It should by now be clear that a casual treatment of the *savoir/connaissance* distinction would be a serious error. On the other hand, finding a solution is not a trivial pursuit. One solution would be simply to transfer the words, untranslated, as we have done with the similarly untranslatable "*Didactique*." On the other hand, that would be cumbersome (witness the paragraph above!) and, given the conjugations of the verbs, both obscure and distracting. Past efforts have included use of "know-how" and "a knowing," but neither has proved very satisfying. We therefore propose in this article to try an idea recently invented by Brousseau: "*Connaître*" derives from the Latin "*cognoscere*" and "*savoir*" from the Latin "*sapere*." We will generally use "to know," "knowledge" and "a piece of knowledge" (the latter for the singular form of either noun), but if there is a need to distinguish, we will pay hommage to the Latin by attaching the prefix "c-" when the word comes from a form of "*cognoscere*" and "s-" when it comes from a form of "*sapere*." We hope in this way to achieve the best available balance of accuracy and readability.

1.2. Brousseau

The teaching sequence presented in this article is the first of a set of six designed for teaching rational numbers and decimals as required in the school curriculum. The first five were used experimentally for around 10 years in two classes each year. With the addition of the sixth part and a review chapter, the sequence covers essentially all of the teaching objectives at both the primary and secondary level. All of the activities organized by the teachers for their students were described and explained, so that they could be reproduced the following year. Eventually the descriptions were assembled into a single book, along with reports of their results and commentaries. In this article we reproduce or summarize the first portion of that book.

These sequences, which were regularly reproduced with 10- or 11-year-old students by the teachers at the École Jules Michelet in Talence, and which could be used with children from 10 to 14 years old, were not designed to be published and used in schools which don't have the benefit of solid mathematical and didactical assistance. Thus they do not represent model lessons, but rather an experiment in *Didactique* and epistemology.

The first three modules from the teaching sequence are summarized below. In them, rational numbers are introduced and studied as measurements and as scalar ratios, with their operations of addition, subtraction, then multiplication and division by a natural number scalar. After being created by the imagination in order to make things commensurable, they are defined in the classical manner by partitions of unity and intermediate units. In the course of the following four modules, students study the ordering of rational numbers, then decimal numbers as a means of rapidly evaluating and comparing rational numbers, the particular properties of operations on decimals, and decimal notation.

The next modules (8, 9) introduce rational numbers as rational linear applications, initially used for drawing reproductions of pictures and geometric similarities. That then makes it possible to give completely general definitions of multiplication and division by a rational (Modules 10–13) and the composition of applications (Modules 14 and 15; fractions of fractions).

It would be useful here to recall a few of the questions which were being addressed in this experiment and a few of the ideas which underlay its realization. We will present just three.

Public opinion in the sixties was exerting pressure for the mathematics taught in schools to resemble as much as possible, and as early as possible, the mathematics practiced and produced by mathematicians. Some even felt that from pre-school to university everything could be taught in a unique "definitive" form. However utopian the idea may appear today, at the time it didn't seem impossible to meet that challenge, or at least to study it seriously.

To do so required that *the activity of mathematicians be modeled*, and then that conditions be imagined which were *realizable by the teacher* and which would lead the students to produce on their own, *by a similar activity*, some current mathematical c-knowledge. In point of fact, there is no such thing as a "mathematical activity" which does not depend on its objective, and the historical genesis of any mathematical concept is so complex and so much wrapped up in its history that it defies reproduction by any isolated modern individual.

Another aspect is that "understanding" a notion like that of rational or decimal number implies that at the end of the learning process a subject has at her disposal a collection of widely varied, logically interlinked pieces of knowledge. This organization can determine an ordering of teaching based on logical relations, for example, a locally or completely axiomatic ordering. Moreover, these are the orders of succession which dictated the classical didactical methods.

But mathematical concepts are constructed in the course of a story which follows another ordering: that of questions, of problems and of solutions, where a much richer collection of "reasons" comes into play. The first idea was thus to realize a process of construction of concepts important to the school curriculum — rationals and decimals — which simulates as well as possible that sort of genesis. That is, to simulate a process making minimal use of pieces of knowledge imported by the teacher for reasons invisible to the students. This type of project was subsequently labeled constructivist.

The initial objective of the experiment was thus an attempt to establish an "existence theorem":

- Would it be possible to produce and discuss such a process?
- Would the students — all of the students — be able to engage in it?
- Could the result of the process be, for each of the students, a state of knowledge *at least equal* to that obtained by known methods?

The realization of the process made no sense unless simultaneously each lesson was conceived, studied, corrected and criticized with the most severe of theoretical, pragmatic and methodological instruments. These instruments were mostly derived from the Theory of Situations, but we consider that they were heavily modified in the course of the experiment. The goal was, thus, that the instruments should progress. *The second objective was to clarify and complete the Theory of Didactical Situations.*

On the other hand, there was no question of relying on imagination and fantasy and then waiting to see if the results were satisfactory. Children are not laboratory animals. The methodological and deontological principles were very different from those in use still today in that domain. In this real experiment, we set minimal objectives in terms of success rates, in terms of median results at other schools, and time limits. If the method in question had not made it possible to achieve the results normally attained by classical methods in that much time, we would have had them follow some specific activities — if necessary using other methods. The comparison between two methods was thus made *on equal results on curricular objectives* by comparing:

- the time and effort required to achieve this result,
- various differences in results which were not evaluated and were often impossible to propose as objectives, of which we will speak later, and
- certain qualitative differences, some of them affective: pleasure and motivation, for the students and the teacher.

The third objective was essentially to know if the use of activities similar to those of mathematicians would give the scholastic c-knowledge of students different qualities from that obtained by the standard teaching methods of the period.

The realization of such a program led the researchers to make a considerable number of "technical" choices about which information and arguments can be quite complex. We will attempt to give an idea of them in the course of our description of the process.

In order to construct such a didactical process it was necessary to determine the knowledge to be taught, its structure, its properties, its different mathematical aspects, its ways of being written and its uses. For our teaching objective we chose the production *by the students* of the principal notions and properties, for reasons of consistency and appropriateness which they themselves could formulate. Numerous pieces of s-knowledge were considered equivalent in the classical methods, even though the results of testing demonstrated that they were not so for most students. The basis for analyzing the situation is to pose the question "Why?": why would the students do this or say that? Why would they change their opinion? Why would they judge it incorrect?, etc. This method led us to decompose this school s-knowledge into distinct pieces of c-knowledge without being preoccupied with restrictions imposed by the classical curriculum. For example, fractions as measurements of size, fractions as scalar ratios and fractions as linear applications are not conceived in the same way or in the same circumstances. Knowledge of one does not guarantee the use of the others, and they should therefore be constructed separately, then brought together in distinct didactical activities.

The terminology used in the text is that used by the teachers. Naturally almost none of those terms would be used with the students in the first five sections. But they are determined and explained by their immediate usage in such a way that no prior mathematical knowledge is necessary. It would be easy — and at times distracting — for a mathematician to find within the sequence of problems and exercises with apparently modest ambitions not only a distinct, separate and ordered study of all the objects in the field (numbers, measures, ratios, fractions, linear applications, homotheties, etc.), of all their structures (algebraic, of order, topological, functional), and of all the properties which together lead to the understanding of the different uses of rationals and decimals, but also properly mathematical procedures of construction (symmetrization, imbedding) which here plainly play the double role of means of construction and demonstration of knowledge. And it would then be apparent to the mathematician how easy it is to put into mathematical format the wealth of knowledge the students have thereby harvested.

The same principle led to a search for "reasons" to study each aspect, either practical or theoretical reasons, so as to base their definition on a genesis and not on an erratic juxtaposition. For example, the reason for inventing both the rational and the decimal numbers is clearly to have the use of a dense set of "numbers" for purposes of measurement and calculation.

But how can we justify the invention of one or the other? Should we follow the historical order? Rational numbers were measurement fractions, ratios of lengths, then ratios of numbers. Decimal numbers are particular rational numbers which were not accepted into mathematics until a very long time after fractions and rational numbers, as a palliative for difficulties of fraction calculation especially in questions of ordering and topology.

But the principle of decimals had been known since the dawn of time as an extension of decimal measures. Their completion generates the real numbers more easily than that of the rationals. The latter could, in fact, be relegated to an appendix: the study of the symmetrization of decimals by multiplication.

This author (G.B.) shares with numerous mathematicians the opinion that it is useless to develop the study of rational numbers in K-12 schools, at least in countries which have developed the decimal system of measurements. On the other hand, decimal numbers are genuinely indispensable today in both mathematics and its applications. It was not possible to draw the didactical conclusions to which this situation obviously points. On the contrary, the study of ratio and proportion which for a long time was maintained as an "elementary" alternative to algebra contributed to the restoration of the study of fractions at the elementary level for reasons of culture or mathematical classiness, and it disappears just when it might have become useful.

It would have seemed natural for this author to conceive of a curriculum which matched his ideas, one *without* the study of rationals, or with the study of rationals as a corollary to that of decimals, in order to propose it to the teachers. But such a teaching sequence, or at least a reasonably similar one, had already been carried out, and according to the claims of the teachers, presented no notable difficulties. An honest defense of the opposite position and the study of its consequences would facilitate if not the invention, then at least the exercise and improvement of the principles of the theory of situations, and would give a "benchmark" to which other methods could be compared.

The designers thus encountered at each step of the process delicate alternatives: should measurement be done by commensuration or partition of unity? Should one start with simple fractions (small denominators) or plunge straight in with any old denominator?, etc. It seemed best to respond with choices which maximized the logical coherence from the point of view of the subjects engaged in the construction, and not as a function of our own knowledge, habits, etc. We felt that in thus augmenting the distance — the transposition — between the reasons natural to the teacher and the reasons, different ones, made natural to the students by the process, we would achieve the clearest view of the constraints of the didactical situations which we wished to study.

The curriculum obtained in the end is composed of 60 lessons, which appears to exceed the time normally allocated to the subject. But in compensation we realized that it integrates the entire of the set of classical arithmetic objectives for the fifth grade, which leaves a lot of time for the teaching of other mathematical subjects.

The last objective is perhaps the most interesting for a reader today. It was a matter of explaining to the teachers and to their teachers the mathematical, epistemological and didactical basis of the concepts they were teaching, and of doing so under conditions and in terms usable by their students, so that this work constitutes a genuine mathematical treatise.

Some will want to know the most notable effects of this teaching. Setting aside the mountains of varied evaluations, just one stands out for me. Students often returned in the following years to show their teacher their grades (and their grades on fractions were always good!). The comment heard more often than any other was, "Yes, I am learning, and I am doing well in my math class, but when are we going to get back to really doing mathematics?"

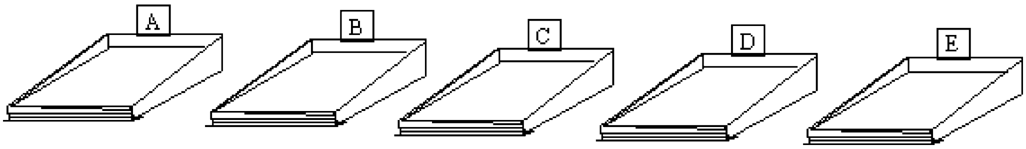

Fig. 1. The five stacks of paper.

2. Module 1[1]

2.1. *Comparison and measurement of the thicknesses of sheets of paper (construction of the rationals — measurement by commensuration)*

The objective of the first module is to define a new way to compare very small lengths, in fact, the thicknesses of sheets of paper. Without the possibility of using their habitual technique, which consists of measuring something with the aid of a smaller unit which one repeats several time, the students "discover" the means, which consists of repeating many times the thickness to be measured in order to compare the result with an appreciably larger unit.

2.2. *Session 1*

2.2.1. *Situation: the thickness of a sheet of paper*

The set up. On a table at the front of the classroom are 5 stacks (or half-boxes) containing 200 sheets each of paper (see Fig. 1). All the paper is of the same color and format, but each box contains paper of a different thickness from the others (e.g., card stock in one, onionskin in another, etc.) The boxes are set up in a random order and labeled A, B, C, D, E. Some of the differences should be impossible to determine by touch alone. The teacher needn't know the exact measurements, since there is no "good measure" to be discovered.

On another table at the back of the classroom there are 5 more stacks or boxes of the same papers, in a different order, which will be used in phase 2.

Each group of five students should have two slide calipers (a device for measuring thickness, standard in French elementary classrooms).

There should be some means of screening the ends of the room from each other — a curtain or a screen or something similar.

2.2.2. *First phase: the search for a code*

The teacher divides the class into teams of four or five students.

Presentation of the situation — assignment. "Look at these sheets of paper that I have set up in the boxes A, B, C, D, E. Within each box all of the sheets have the same thickness, but from one box to another the thickness may vary. Can you feel the differences?"

Some sheets from each box circulate, so that the students can touch them and compare them.

"How do businesses distinguish between types?" (weight)

[1] An earlier version of this section appeared in Brousseau (1997). Both are based on portions of Brousseau, N. & Brousseau, G. (1987).

"You are going to try to invent another method to designate and recognize these different types of paper, and to distinguish them entirely by their thickness. You are grouped in teams. Each team must try to find a way of designating the thicknesses of the sheets. As soon as you have found a way, you will try it out in a communication game. You may experiment with the paper and these rulers."

Development and remarks. The students almost invariably start by trying to measure a single sheet of paper in order to obtain an immediate solution to the assignment. This results in comments to the effect that "It's way too thin, a sheet has no thickness" or "it's much less than a millimeter" or "you can't measure one sheet !"

At this point there is frequently a moment of disarray or even discouragement for the students. Then they ask the teacher if they can take a bunch of sheets. Very quickly then they make trial measurements with 5 sheets, 10 sheets — until they have a thickness sufficient to be measured with a ruler. Then they set up systems of designation such as:

10 sheets 1 mm

60 sheets 7 mm or

31 = 2 mm[2]...

In this phase, the instructor intervenes as little as possible. He makes comments only if he observes that the students are not following — or have simply forgotten — the assignment. The students are allowed to move around, get more paper, change papers, etc.

When most of the groups have found a system of designation (and the five children in each group agree to the system or code) or when time runs out, the teacher proceeds to the next phase: the communication game — going on even if not every group has found a system.

2.2.3. Second phase: communication game (ca. 15 min)

Presentation of the situation — assignment. "To test the code you just found, you are going to play a communication game. In the course of the game you will see whether the system you just invented actually permits you to recognize the type of sheet designated."

Development. "Students on each team are to separate themselves into two groups: one group of transmitters (two students) and one of receivers (two or three students). All the groups of receivers go to one side of the curtain, and the groups of transmitters to the other. The transmitters are to choose one of the types of paper on the original table, which the receivers can't see, thanks to the curtain. They will send their receivers a message which should permit the receivers to find the type of paper chosen. The receivers should use the boxes of paper set out on the second table at the back of the classroom to find the type of paper chosen by the transmitters.

When the receivers have found the paper, they become transmitters (after verification with the transmitters). Points will be given to the teams whose receivers have correctly found the type of paper chosen by the transmitters".

At the beginning of the game, the teacher puts the curtain in place. Then he:

- passes the messages from the transmitters to the receivers,
- receives the responses of the receivers,
- checks whether this response corresponds to the choice of the transmitters and announces the success or failure to all of the team.

[2] This use of the equal sign is incorrect. The teacher will mention it during the discussion time.

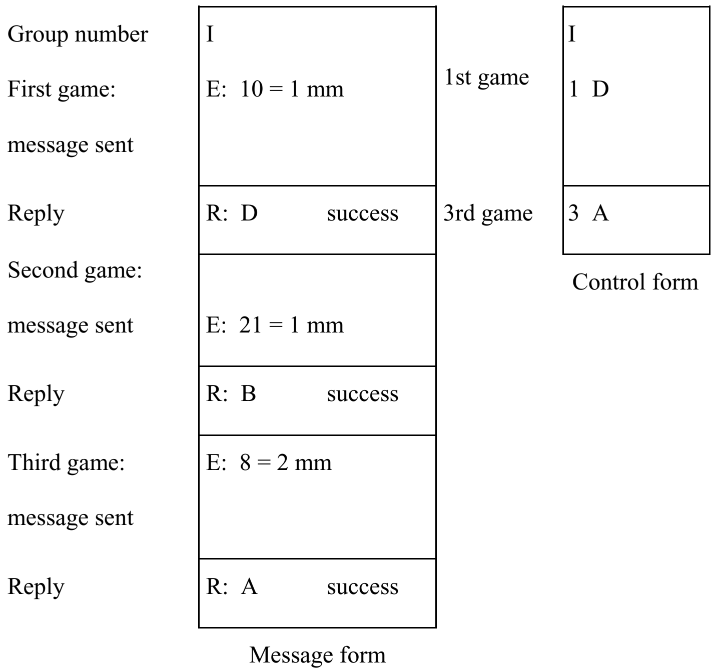

Fig. 2. The message card.

All of the messages are written on the same sheet of paper, which we can call the “message card” (see Fig. 2), which goes back and forth between the transmitters and receivers on the same team. The team’s number is written on the card. In addition, the transmitters write on another sheet of paper — the “checking card” — which they keep, the type of paper which they have chosen, so that the teacher can check for success or failure.

Remark. Clearly, the teacher does not introduce superfluous formalism or vocabulary. If certain teams have not arrived at any way of sending effective messages, the teacher could send them back to considering a code together (same assignment as in the first phase). On the other hand, in eight identical trials of this material, that has never happened. The students have managed to play two or three rounds of the game.

Behaviors. During this game, there are three different strategies commonly observed.

- Some choose a particular number of sheets and always measure that.
- Some choose a particular thickness and count how many sheets it takes to make that.
- Some look randomly at a thickness and a number of sheets.

It is notable that the children prefer to choose the types of sheets of extremes of thickness: the thinnest or the thickest, to make the job of their partners easier.

2.2.4. Third phase — result of the games and comparison of the codes (20–25 min)

Presentation of the situation and assignment. For this phase, the students go back to their original places in teams of 5, as for Phase 1. The teacher prepares a table with groups down the side and paper types across the top and keeps a record of the groups’ messages (and their success) as the reports are made.

Development and remarks. Taking turns, each team sends a “representative” who reads the messages out loud, explains the code chosen and indicates the result of the game.

The different messages are compared and discussed by the students. Since they are frequently very different, the teacher requests that they choose a common code.

Example: 10; 1 mm
VT (for Very Thin)
60; 7 mm

After discussing these, the class chose 10; 1 mm and 60; 7 mm.

When all the messages have been written up, the students inspect the table and make spontaneous observations like "That doesn't work!" and "That one's OK," etc. These remarks fall into four categories.

1st category

If the sheets are of different types, the same number of sheets should correspond to different thicknesses.

Example: 19 s, 3 mm - Type A
19 s, 3 mm - Type B
} "That doesn't work!"

2nd category

If the sheets are of the same type, the same number of sheets should correspond to the same thickness.

Example: 30 s, 2 mm - Type C
30 s, 3 mm - Type C
} "That doesn't work!"

3rd category

If there are twice as many sheets, it should be twice as thick.and the students add: "it should be"

Example: 30 s , 3 mm - Type C
15 s , 1 mm - Type C
} "That doesn't work!"

because 30 s ; 2 mm ↓ x2 15 s ; 1 mm and 15 f , 1 mm ↓ x2 30 f , 2 mm

4th category

A difference in the number of sheets shouldn't correspond to the same difference in thickness.

Example: 19 s , 3 mm
20 s , 4 mm
} "That doesn't work, because one sheet can't be a millimeter thick!"

At the end of the session the teacher proposes to the students that they finish up the table the next day by collectively verifying the measurements and fixing them if necessary.

Important remark. The use of arrows to indicate operations carried out on the numbers in the process of finding equivalent pairs has no formal or obligatory character. It is a familiar "manifestation" of the use of natural operators to which the students are accustomed. The teacher makes no explicit reference, and does not ask of the students any explicit reference, to the scholastic s-knowledge attached to the c-knowledge of proportionality. On the contrary, she favors the explanations given by the students to whatever extent they are understood, but does not at this stage correct the ones which are not understood.

2.2.5. Didactical results

All of the students know:

1. how to measure the thickness of a certain number of sheets of paper (with or without the calipers),
2. how to write the corresponding ordered pair, and
3. how to reject a type of paper which does not correspond to a written notation given to them (if the difference is large enough).

   Most of them are thus able:

4. to analyze a table of measurements, and
5. to point out inconsistencies making *implicit* use of the linear model.

   Those who can't do so seem to have understood those who do.

   This c-knowledge is sufficient to undertake (understand the goal and resolve) the situations which follow, where the need is:

6. to distinguish between *numbers for counting* and *numbers for measuring* the thicknesses, and
7. to use these to carry out the additive operations: addition, repeated addition, subtraction.

The remaining two sections of the first module explore some of the consequences of the discoveries made in the first session and introduce the standard fraction notation.

2.3. Comparison of thicknesses and equivalent pairs

The first step is a review of the table produced in the previous lesson. At first, students study the table silently and make individual observations; then they discuss these observations as a class. The table is corrected either by universal agreement, or, where that agreement doesn't occur, by a re-measurement. This process serves to bring out the idea of augmenting the number of sheets counted in order to distinguish between papers of highly similar thicknesses as well as to exercise further the implicit use of linearity to determine consistency of representations of the same paper.

Working in (non-competitive) groups, students then fill in any empty slots on the chart by counting sheets and then comparing their results with those of other groups. As a confirmation and celebration, they play one more round of the communication game from the first session, discovering that they are now equipped to handle it even if a couple more types of paper are tossed in. This finishes the second session.

2.3.1. Equivalence classes — rational numbers

In the third session of Module 1, the completed table is once more the center of attention, and the central topics are equivalence and comparison. After getting the students to focus on the table, the teacher presents some other pairs of numbers and asks which kind of paper each pair represents, then has the students invent other representations, listing all of the accepted ones in the same column on the table. This provides the occasion for introducing the term *equivalent*.

The teacher then produces a new table with a single representation for each kind of paper (the class chooses the representation) and the students are told to figure out the order of the papers, from thinnest to thickest. Students work individually, then discuss their results and their reasoning. Once an order is agreed on, the teacher introduces another type of paper (fictional this time) and the students figure out where in the ordering it belongs.

As a final step, the teacher returns to the table with columns containing equivalent ordered pairs for each type of paper and introduces the word *fraction* and the standard notation for a fraction, pointing out that this not only makes it possible to designate the entire class of equivalent pairs, but also gives a designation for the thickness of a single sheet of paper.

The lesson finishes with some opportunities for the students to practice the use of this new notation and its connection with papers.

3. Module 2

The next five lessons constitute the second module, which deals with operations in the context of the sheets of paper.

3.1. The thickness of cardboard

By way of motivation for introducing operations, the teacher asks students to consider individually and then discuss together the issue of whether the “rational thicknesses” they invented in the previous lessons are numbers. In general the conclusion is that if you have 8/100 the 8 and the 100 are numbers, but 8/100 is two numbers. The teacher points out that we might be able to regard them as numbers if we could do the same things with them that we do with numbers, and asks what those things are. Responses generally include “count objects with them,” “put them in order” and “do operations like addition and multiplication with them.” Quietly slipping the first of these under the carpet, the teacher presents the suggestion that to decide whether these are numbers they need to try to do some operations with them.

The first project is to make “cardboard” by sticking together a sheet of type A paper (thickness 10/50) and a sheet of type B paper (thickness 40/100). “How thick do you think the resulting pages will be?” Students agree that that thickness will be $10/50 + 40/100$, and most agree that the result will be 50/150, though a few have some doubts about that. After a short discussion, whatever its outcome, they set out to verify the results. The teacher has them count out 50 A sheets and 100 B sheets and begins gluing them in pairs, continuing until students realize that a problem is developing and stop the process. Offered an opportunity to correct their proposed solutions, most go immediately to the correct procedure. Most are, in fact, so confident that they declare verification unnecessary, but the teacher does it for the sake of the others. The stack may measure 59 or 61 mm, but this they have already learned to deal with. They then practice by adding some other pairs and triples of fractions, and observe that they are now capable of adding any fractions they want.

The remarks on this lesson have a wide enough application to be worth reproducing in their entirety:

The choice of thickness to add could be anything, but the manipulations depend on the numbers. For example, one could propose $10/50 + 39/100$. This would prevent the children from mixing two boxes of paper, but not from envisaging it and saying that it is impossible.

To offer at this particular moment the sum of two fractions with like denominators would be a didactical error. Certain teachers have tried it with the hope of obtaining an immediate success for everyone. They wanted to avoid having students have the double difficulty of having to decide to reduce to the same number of sheets and doing it in such a way that the sum of the numerators, that is, the thicknesses, would make sense. Doing so gives the children justifications which are easy to formulate and learn, which facilitates the formal learning of the sum of two fractions (we know how to add two fractions whose denominator is the same, so what is left for us to do in the general case is to reduce it to having the same denominators before performing this addition).

But this method gives inferior results. Only the students capable of comprehending simultaneously and immediately both the general case and the reasons for the apparent ease of the particular cases were able to avoid difficulties in developing a correct concept of the sum of two fractions. They were then able to reason directly or make rapid mental calculations (for example, $15(3/100 + 7/15)$). The rest were distracted from the pertinent questions (such as why the denominators can't be added) and the efforts necessary to conceive of and validate the concepts by the apparent ease of carrying out the action. They were invited to learn a method in two stages, with the possibility of some false justifications for the first stage (if I add 3 hundredths and 5 hundredths that makes 8 hundredths, just the way 3 chairs and 5 chairs make 8 chairs). They first *learn* that it is possible to add fractions which have the same denominator, and how to do it. They also learn that it is not to be done, or can't be done, if the denominators are different (you can't add cabbages and wolves!). Then they *learn* to solve the other cases by turning them into the first case, not because of the meaning of this transformation, but because it works. The economy of this process is strictly an illusion, because there is no representation to support the memorization. It will furthermore require a large number of formal exercises to make the process stick and to make it possible to distinguish it from other calculations. Some students never do get it figured out.

Using different denominators, on the other hand, all the children are able to come up with the concept and solidify their representations with experimentation and verification in a way that makes any formal teaching unnecessary. A delay in algorithmisation can at times be of considerable benefit to conceptualization.

3.2. What should we know now?

The next session comes in two sections which look similar but have quite different functions. Each contains a series of problems. Those in the first section are designed to let the children make use of what they have figured out in the first session, both in order to solidify that knowledge and to extend the range of mathematical activities it can be used for. The first problems are strictly review. The teacher writes up several pairs or trios of fractions to add, walks the class through the first one, speaking in terms of thicknesses of the two papers, and turns them loose on the rest. The next problem is to find the thickness of a sheet obtained by gluing together one of thickness 4/25, one of 18/100 and one of 7/50. Following that, they work on $8/45 + 5/30$. The last in this set returns to asking the question in terms of the sheets themselves: "A woodworker is making a collage for a piece of furniture. He glues together three pieces of wood of different thicknesses: 40/50 mm, 5/25 mm and 6/10 mm. List these woods in order of thickness, then say how thick the resulting sheet will be."

In each case, the problem or problems are to be solved individually, then presented to the class for discussion and validation. Included in the discussion is the possibility of having several correct routes to the same solution.

Remarks at the end of this section of the session emphasize its status as well as how it is to be carried out.

The object of this phase is to permit the children to make use of the procedures they discovered in the previous session, to generalize them and make them more efficient. That is, to let them evolve.

This session is thus neither a drill nor an assessment. Also the teacher is not to pass judgment on the value of the methods used, nor at any moment to say which solution is correct.

For each exercise, she organizes and facilitates the following process:

- Individual effort
- Collection of results
- Comparison of methods
- Discussion and *validation by the students*

A method is accepted if it gives a correct solution (in that case an "acknowledged" and correct method), rejected if not. Among the methods which have been accepted, remarks on length or facility of execution, which the teacher solicits, do not become judgments of value which the child can confuse with judgments of validity. On the contrary, the teacher sees to it that the child takes part in the debate, has a result to offer, is able to discuss his methods and state his position relative to his own knowledge.

The immediate collective correction and rapid discussion of the problems is thus indispensable. It enables the teacher and everyone else to know the stage of assimilation of each child and what she is having difficulty with. The whole class can take part in each student's effort.

The second phase of the session is a set of individual exercises for drill and assessment. It has a classic didactical form: written questions to be answered individually and turned in for correction (outside of class) by the teacher. The problems represent each of the levels of operation with fractions thus far obtained — ordering of fractions with unlike denominators, addition of fractions with denominators which are like, or one of which is a factor of another, or which require a common multiple.

This frequently results in some rather poor papers, especially since part of its function is to accustom students to the (as yet) unfamiliar task of producing mathematics for which they have no immediate feedback.

3.3. The difference of two thicknesses

The next session proceeds to the subtraction of two thicknesses. It requires more types of paper, with thickness ranging up to that of heavy card stock, but only one sheet of each of them (for demonstration purposes).

The lesson starts with a rapid discussion of the problems handed in the day before. Only the ones where errors were made need be mentioned, and the teacher needs to restrain herself firmly from letting the discussion of the common denominator in the last problem lead to one of the methods taking on the status of Official Method.

The next stage begins with a swift return to the initial situations: what does 8/50 mean? (the thickness of a sheet of paper such that you have to have a stack of 50 of them to measure 8 mm). And what does $8/50 + 6/100$ mean? (the thickness of a sheet made by gluing together an 8/50 thick sheet and a 3/50 thick sheet).

Remark. It is often useful to insert a reminder like that of preceding situations, for two essential reasons:

- In the first place to allow children who have some difficulties or are little slow to be more thoroughly involved in the present lesson.

- Furthermore to allow children who have been absent to understand what happened in the previous lessons and be able to participate in the following one.

The teacher then writes on the board

$$8/50 - 6/100$$

and asks the class what that might mean and how to carry out what it says to do.

This launches a discussion which starts with a predictable set of misinterpretations and arrives fairly swiftly at the realization that it is the card stock which is the very thick one, and it is made up of the thin one glued to one of unknown thickness. With a drawing on the board to represent this and the equation $6/100 + \square = 8/50$ beside it, the students are turned loose to work individually on finding, “by trial and error or calculation, for example” just what that thickness might be, and how to verify their results.

The resulting discussion includes many variations, a number of them correct. Students who have not succeeded give the results they got and say whether they are too large or too small.

Next the class solves (and interprets at the same time) $4/15 - 1/15$.

The problem $4/50 - 3/40$ is launched by getting the class to state the need for a common denominator, then left for individual work. Then for a final problem, worked individually, they take on $12/8 - 2/5$.

3.4. Thickness of a fat piece of cardboard: product of a rational number and a whole number

This lesson requires 10 sheets each of four highly distinguishable types of paper, each with a known thickness. Students are set up in groups and each group is assigned a single type of paper. They are to determine the thickness of a sheet made up by gluing 3 sheets of their own paper together, then 5 sheets, then 20, 100 and 120. Each group figures out all of their own, then writes the results on the blackboard. Each group then checks one other group’s results and either signifies agreement or supplies an alternate answer. Enough students are solidly in control of the material so that the ensuing discussion produces a general agreement, and the table of values can be successfully corrected.

The final phase of this session is a comparison of the thickness of the various cardboards with 1 mm. The teacher chooses one of the thicknesses in the table, for instance 57/35, and asks the students whether they have any idea how thick that card really is. Is it thicker or thinner than 1 mm, or equal to it?

In groups of two or three, students set to work. A lot of them take out their rulers to have a more precise idea of a millimeter. Some work out elaborate approximations, many point out that 35 sheets would make up exactly a millimeter, so 57 of them must be thicker than that (“but not 2 mm thick!”) and a few are completely bewildered. After a certain amount of discussion of this particular thickness, the assignment becomes: “Look at the table and see what else you can say about the thicknesses.” This gives rise to a lively discussion and a lot of joy in discovery.

Remark. This last part proceeds informally and spontaneously for the pleasure of exchanging and discussing ideas without any pressure from the teacher. The teacher listens to the remarks and says nothing unless the students ask him to clarify or explain something.

It is essential to insist on the fact that the teacher has not set out any contract of learning or acquisition. Some children may take the analysis of the situation a huge distance and make subtle, profound remarks. Others have intuitions which they are unable to communicate. These “discoveries” meander a bit, but it doesn’t matter — the jubilation of the ones who have found something wins over the ones who listen, approve, look at them in incomprehension or contradict them. Anyone can advance a notion or even say something “dumb.”

The teacher restricts himself to making sure people take turns, without interfering with the order or the choice of speakers, in order to maintain the group’s pleasure in this game. To do that, he has to register his own pleasure, but make sure that his pleasure is not the children’s goal.

He takes note of errors and difficulties without trying to correct them right away. If no one notices them, then in general an explanation at that point would do no good. The teacher has to consider it as an obstacle which needs to be taken up later in a prepared didactical activity. But frequently after a moment a student notices the error and the debate revives. Obviously, it has to be clear that the teacher’s silence doesn’t indicate either acceptance or rejection. And it’s not enough to *say* it — he has to *do* it.

3.5. Calculation of the thickness of one sheet: division of a rational number by a whole number

First the students remind themselves how to multiply by figuring the results of gluing together 5 sheets each 3/9 of a millimeter thick. Then they are presented with: “I’ve glued 9 equally thick sheets of paper together and the resulting card is 18/7 mm thick. What could we ask about it? (the thickness of each sheet). Can you figure out the thickness? If so, write it in your notebook.”

Individual work very swiftly produces the correct result and reasoning. Also the idea of writing the problem as $18/7 \div 9$. This requires a little delicacy in handling, since they only know for sure that division is defined between whole numbers, but the idea certainly needs confirming, especially after students observe that the operation here can be successfully inverted with a multiplication by 9. The major point to emphasize is that it is the whole fraction (the thickness) which is to be divided, not just the numerator or denominator. This becomes clearer with the next situation: “Now I’ve glued 9 other equally thick sheets together and made a new card. This one is 12/7 mm thick. Can you find the thickness of each of the sheets I glued together?”

Students work in groups of 2 or 3, then share their results. Since two of the most accessible solutions are multiplication of the original fraction by 3/3 and 9/9, the resulting discussion is likely to include a brief furor until somebody observes the equivalence of 12/63 and 4/21.

The final activity is to work individually on $(13/5) \div 9$, first giving it a meaning, then calculating the result. Students tend to bypass the former and work on the latter, which means the teacher has to lean on them to write the sentence in question. After 5 min or so, the teacher stops the work and sends one or more students to the board to write up their solutions. By and large they multiply by 9/9 and then divide the numerator by 9. Only occasionally does somebody observe that the denominator has just been multiplied by 9, and the level of generality of this observation remains undiscovered.

3.6. Assessment

The module finishes with a set of problems for a summative evaluation.

4. Module 3

The third module extends the students’ thinking beyond sheets of paper, with the objective of giving them enough similar experiences to make generalization plausible and legitimate.

4.1. Fractional measure of weight, capacity and length

The lesson requires a considerable collection of materials:

8. To measure weight, a balance beam and five different categories of nails,
9. To measure capacity, five small glasses of different sizes, one colored glass to serve as a unit and two (largish) test tubes, one of them with a sticker on it so that they can be distinguished, and
10 To measure length, strips of construction paper of equal width but different lengths, a single strip of gray cardboard (same width, yet another length) to serve as the unit and a big piece of poster paper to work on.

The glasses and the strip lengths need to be chosen in such a way that none is an integer multiple or divisor of the unit. Seven nails of one sort have the same mass (balance on a scale) as eleven of another. If the first serves as a unit, the second weighs 7/11 unit. The contents of three "unit" glasses emptied into one tube comes to the same height as the contents of five glasses A emptied into the matched tube. Glass A holds 3/5 of a unit.

The class is divided into groups, each responsible for only one category of measurement. Each group proceeds as in the first session of Module 1 — decides as a group on a "code" to denote a particular type of nail or a particular glass or paper strip. They then repeat the communication game (choreographed this time in such a way that there is no need for additional sets of nails, etc.!) The codes used in the game are listed on the board (together with their success or failure) and each is explained by the group that invented it.

The class concludes that fractions can be used to measure weight, capacity, etc. The session finishes with some practice questions, e.g., "What does it mean that this glass has a capacity of 3/4 of the unit?," "If I stick together a nail weighing 17/25 of the unit and one weighing 40/75 of the unit, how much will the resulting object weigh?"

4.2. Construction of fractional lengths (a return to the classical conception)

This session is one which builds a clear (albeit temporary) bridge between this lesson sequence and the more familiar ones. We have therefore chosen to reproduce it with minor alterations rather than summarize it.

In the previous session, the children attached numbers to sizes (they designated a measurement). In this session, they will construct objects whose measurement in terms of a unit is given (they will realize a size). We deal only with lengths for material reasons.

This construction will suggest a technique based on a new representation of the notion of fraction.

To represent a strip of length 5/4 of the unit, the representation already introduced permits several methods. Either one takes a random length, repeats it 4 times, compares the result with the length of 5 units and then corrects by trial and error, or one repeats the unit 5 times and divides this length in 4. This method requires the student already to have a fairly flexible ability to use the definition given.

There is a third, more efficient method if one wishes to construct a number of strips with denominator 4: divide the unit in four and repeat the resulting piece five times. This is the method we are going to try to induce, without hoping that the children will show or say that it is equivalent to the other.

4.2.1. Materials

Note: all the strips are the same width, around 2 cm, and are made of construction paper.

- 12 unit strips (gray) 20 cm;
- 4 identical sets of 6 strips (green) whose lengths are respectively

  5 cm (1/4 unit), 10 cm (1/2 or 2/4 unit), 15 cm (3/4 unit), 30 cm (3/2 or 6/4 unit), 35 cm (7/4 unit), and 45 cm (9/4 unit);

- 4 identical sets of 6 strips (blue) whose lengths are respectively
- 4 cm (1/5 unit), 8 cm (2/5 unit), 16 cm (4/5 unit), 24 cm (6/5 unit), 28 cm (7/5 unit), and 36 cm (9/5 unit);
- 4 identical sets of 6 strips (yellow) whose lengths are respectively

  2.5 cm (1/8 unit), 5 cm (2/8 unit), 12.5 cm (5/8 unit), 17.5 cm (7/8 unit), 22.5 cm (9/8 unit), 27.5 cm (11/8 unit);

- scissors;
- strips of poster paper 50 cm long and 5 cm wide; and
- long strips of construction paper, all 2 cm wide.

4.2.2. Phase 1: communication game

The class is divided into 12 groups of 2 or 3 children. Each group has 1 unit strip and 1 set of 6 strips of the same color.

Assignment. "Each group is to find fractions representing the lengths of their six colored strips using the (gray) unit strip and write all of them on the same message pad. So each group starts off as a message-sender.

Each group will receive a message from another group. At that point you all become message-receivers. You are to cut strips of white paper in the six lengths indicated on your message.

Next, each receiver-group will meet with the group that sent the message they decoded and verify together (by superposition) that the white paper strips are indeed identical to the ones used to produce the message. If they are identical, the message-senders are winners.

If you need extra materials (paper, scissors, . . .) they are available".

Development: measurement phase. Children may use the habitual technique represented in Fig. 3a.

Some may notice that of the six strips, five are multiples of the smallest. That means they need only measure that one. Further, when they repeat that little one they can observe that it fits an exact number of times into the unit. They can then use it as an intermediate unit (see Fig. 3b).

Development: communication phase. For convenience, it is the teacher who passes the messages. Groups need to receive messages from other groups whose strips are of a different color (and hence a different set of lengths). Strips of white paper and scissors are given out at the same time as the message.Even if they started out measuring other lengths, this choice of strip-lengths encourages useful observations on the part of the ones who can use them to save time.

4.2.3. Phase 2: report on the results

Students come to the board to present their messages and indicate how they figured out lengths and how they made their constructions. The teacher resists giving personal judgments on the methods used. She restricts herself to making sure the children give clear descriptions of their methods and results.

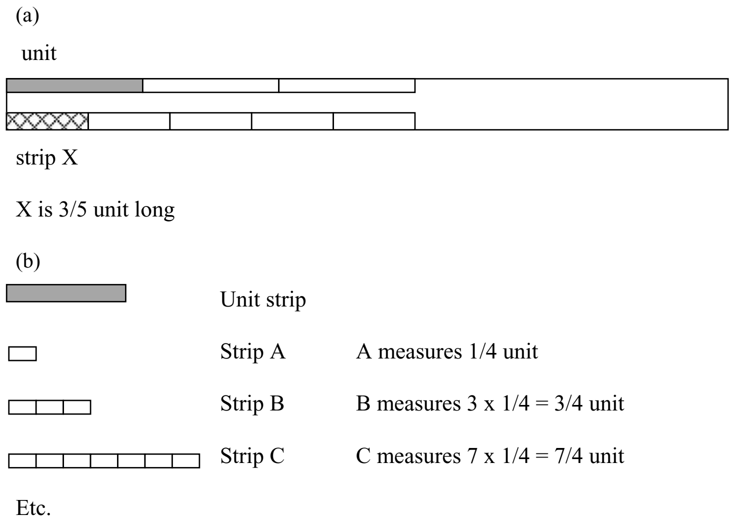

Fig. 3. Representing (a) 3/5, and (b) 1/4, 3/4 and 7/4.

4.3. Comparison of strategies

This session begins with a follow-up discussion in which by use of the solutions written on the board by the children and a process of observations (by students) and (student-proposed) verifications the teacher guides the class to a conviction that this method of "intermediate units" provides a general solution.

The follow-up is a pair of problems to be worked on individually and then discussed.

- A cloth merchant sells first half of a piece of velvet cloth and then a quarter of the same piece. What fraction of the piece is left at the end of the day? The piece was originally 24 m long. What is the length of the remaining piece?
- Claude has a bag of marbles. In the course of a game he loses first 2/3 of his marbles and then another 2/9 of his marbles. What fraction of his marbles has he lost? What fraction of his marbles does he still have? At the beginning of the game, he had 63 marbles in his bag. How many does he have at the end of the game?

The second problem is very difficult for them, both because it is not recognizable as being any of the meanings of fraction they have previously encountered and because they have trouble understanding that the unit here is the bag of marbles. They need help on this one.

Is this difficulty an indication that ratios between discrete quantities are genuinely more complex for the students, as we had deduced from our preparatory studies of ergonomy and of various psychological works, or is it entirely due to the didactical option which caused us to choose to introduce ratios as a means of measuring "continuous" quantities? We think that ratios are indispensable in the measurement of continuous quantities and that their application in discrete measurements (with natural numbers) is a reification[3].

[3] By way of clarification of that comment: a classical order for introducing ratios consists: (1) of showing fractions metaphorically: cakes cut in four (never in seven), of which one takes three parts; (2) of using this "definition" to carry out problems of

In the process of which the three modules summarized here are a part, it wasn't until the end of the students' work on embedding the natural numbers and decimal in the rationals (Modules 4–7) that they had available the notion of ratio in discrete measurements as well as continuous ones.

5. Conclusion

The classic pattern for lessons on fractions is almost invariably the following: just as soon as the formal manipulation of a concept or operation or use has been introduced and explained in some particular situation there is a rush to present the students with all the other formally similar uses as "applications" or even "re-labeling" of the same idea. Thus as soon as *a* division has been introduced in the natural whole numbers it becomes for the teachers *the* division, no matter what the circumstances and the structures are in which "it" is being used, and even if its properties have been drastically modified without the students' knowledge.

The same could happen here. The few lessons which we have just presented would make it possible to handle almost every formal operation on fractions. To be sure, they are a bit different from the standard lessons, but the knowledge they produce is pretty comparable. One could thus imagine that the course could stop there and that it would suffice for exploring the entire universe of classic problems on fractions and rational numbers. These lessons could be followed up by the classical introduction of decimal numbers, which consists of extending the number system with the aid of decimal fractions, but only in order to express measures less than unity. But it seems to us that the didactical procedure of "analogical" or "metaphoric" extension of the use of knowledge is unworthy of mathematics. Is it really necessary to cheat the students on this point? Is it really necessary to make them carry the responsibility of not understanding as identical a bunch of objects which really are very different, but which we wish to confuse? Is it really necessary to declare that what is simply familiar to us should be logically clear to them?

We didn't think so. And that is why we wanted to look into each different mathematical aspect of the uses of fractions and rationals made in the required curriculum, treating each one as a specific problem. Mathematics cannot be reduced to a sequence of algorithms and definitions to be "applied." Mathematics consists of answers to questions, of opportunities to pose new questions and an art of arranging those questions and those answers so as to support the use and learning of them.

This is what we hope to demonstrate in articles to appear as sequels to this one.

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the "rule of three": "A household spends three quarters of its income on domestic expenses . . . ." The students divide by four and then multiply by three, the teachers selects numbers for which that works, especially if the multiplication is done first. In this way the introduction of rationals consists entirely of reformulating in terms of ratios or fractions a bunch of mathematical objects which are already completely known, and that without any apparent reason. We can say it thus, and then thus . . . . But in these conditions, since everything can be counted, there is no need of ratios. On the other hand, the measurement of continuous quantities is essentially tied to the notion of a ratio between a quantity and a unit which is necessarily arbitrary.