On the Role of Constructivism in Mathematical Epistemology

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> Context • the position of pure and applied mathematics in the epistemic conflict between realism and relativism. **> Problem** • To investigate the change in the status of mathematical knowledge over historical time: specifically, the shift from a realist epistemology to a relativist epistemology. **> Method** • Two examples are discussed: geometry and number theory. It is demonstrated how the initially realist epistemic framework – with mathematics situated in a platonic ideal reality from where it governs our physical world – became untenable, with the advent of non-Euclidean geometry and the increasing abstraction of the number concept. **> Results** • Radical constructivism offers an alternative relativist epistemology, where mathematical knowledge is constructed by the individual knower in a context of an axiomatic base and subject items chosen at her discretion, for the purpose of modelling some part of her personal experiential world. Thus it can be expedient to view the practice of mathematics as a game, played by mathematicians according to agreed-upon rules. **> Constructivist content** • The role played by constructivism in the formulation of mathematics is discussed. This is illustrated by the historical transition from a classical (platonic) view of mathematics, as having an objective existence of its own in the "realm of ideal forms," to the now widely accepted modern view where one has a wide freedom to construct mathematical theories to model various parts of one's experiential world.
> Key words • **Knowledge construction, noncognitive knowledge, realism, relativism, platonic world, physical world, non-Euclidean geometry.**

1. What kind of constructivism?

The term "constructivism" has become an important referent for the philosophy of learning and knowledge. In the literature, one frequently finds references to "constructivist learning" and "constructivist teaching," and also to such notions as "a constructivist education" and "a constructivist view of knowledge," etc. However, when scanning the literature in this field, one quickly discovers that there are a great number of different constructivisms around – i.e., many disparate uses of the word, signalling the difference in approach taken by the authors involved. Thus, for instance, one author (Geelan 1997) lists some twenty different forms that can be found in the literature. These forms are then qualified by prefix labels such as: cognitive, contextual, critical, dialectical, empirical, humanistic, information-processing, methodological, moderate, Piagetian, post-epistemological, pragmatic, radical, rational, realist, social, sociocultural, sociohistorical, trivial.

This diversity and its implications are discussed in some detail elsewhere (Quale 2008). From one point of view, it serves to demonstrate the power and fertility of the notion of constructivism. But it also shows very clearly the wide divergence of the research approaches being adopted in this field. Clearly, when engaging in a discussion involving constructivism, it is necessary to state up front which particular kind of constructivist theory one is talking about.

The theory that will be discussed in this paper is known as *radical constructivism* (RC), which was originally proposed, and has been extensively developed and discussed, by Ernst von Glasersfeld (1981, 1984, 1989, 1991, 1995, 2000). It is a fact that this theory has become quite controversial, creating considerable discussion and heated argument in the academic discourse that addresses the philosophy of science.

Glasersfeld (1995) defines radical constructivism in the form of two basic propositions, which may be summarized as follows:

> *RC1* – Knowledge is not passively received, but is actively constructed by the knower.

> *RC2* – The function of this process of construction is adaptive, and serves the knower's organisation of her own experiential world, not the discovery of an objective reality.

We note that proposition RC1 would be acceptable to any theory that purports to be "constructivist"; taken by itself, this proposition is often said to define a theory of *trivial constructivism*. (This appellation may be somewhat unfortunate; the idea that knowledge of any kind is constructed by the knower, and not "downloaded" from the environment in some objective sense, is far from trivial. A better name for RC1 would perhaps be "minimal constructivism," since it expresses the one common element that is shared by all constructivist theories. Still, the name "trivial constructivism" seems to be established in the literature, so I will stay with it.)

On the other hand, RC2 is specific to radical constructivism. This is the proposition that has made the theory controversial: charges have been made by critics that it "denies reality," or at least rejects the possibility of attaining a true description of the real world! Such issues have been discussed at length elsewhere (Quale 2008); here, we will just state some conclusions of this discussion, with relevance for the theme of the present paper.

First, we remark that proposition RC2 highlights the distinction made in this theory between *epistemology* (dealing with the nature and validation of knowledge) and *ontology* (dealing with existence, or being: in this case, the question of whether there exists an objective reality, which is independent of our perception). And here it is important to note that radical constructivism does *not* assert that such a reality does not exist. On the contrary, it accepts as a given that we all inhabit and share a common environment, and that we can interact with each other in this environment. Thus the theory takes a firm stand against the position of *solipsism*: the assumption that any individual knower must resort to inventing her own reality, in whatever way she fancies. For the individual knower, such a solipsist stance may well be *logically irrefutable* – i.e., she can never have a 100% guarantee that the world she perceives around her is not just a hallucination in her mind. But it is also *existentially irrelevant* for her – i.e., she will choose to disregard this possibility in the way she conducts her life. It is in our nature as human beings to assume that the world that each of us experiences is in fact there for us to experience – indeed, to call this assumption seriously into question would generally be considered a sign of mental aberration! Thus, we reject the "solipsist fallacy."

However, this does not mean that we have to retreat to the assumption of an *objective reality*. Radical constructivism asserts that there can be, for any individual person, only one meaningful notion of the term "reality:" namely, *the totality of everything that can be experienced or imagined by that person* – "the world," in the terminology of the philosopher Ludwig Wittgenstein.[1] In fact, radical constructivists will often refer to this totality as the knower's *experiential world*, avoiding the use of the term "reality" because of its unfortunate connotations of "objectivity." It is this experiential world that every individual knower is living in, and continually adapting to through the complex and life-long process of learning. Or, put another way: the experiential world is the only available source of any knowledge that a knower will ever have; and such knowledge can then only be acquired (learnt) by her through a process of personal construction, based on individual perception and reflection. (For later reference, we note here that this conception of an experiential world includes not only the knower's perception of external sense-data and reflection on these but also her knowledge of mental constructs that are not based on external sensual stimuli – for instance, elements of mathematical theory.)

Thus, radical constructivism does not preclude the possibility of an objective reality, existing independently of all knowers. But it does assert that this issue is a matter of preferred belief for the individual knower, and as such *not within the scope of cognitive argumentation*.[2] In other words, it is in principle not possible to obtain cognitive (or, as some would put it, rational) knowledge of such an entity; whatever knowledge we can obtain of it will by definition be *noncognitive*. This distinction between cognitive and noncognitive knowledge is important in the discussion that follows below. However, it should be noted that the term "cognitive knowledge" is used with somewhat different meanings in the literature; in the present paper, it may be described as "intersubjective knowledge, derived from rules of reasoning, that can be shared and agreed on by individual knowers." (I am grateful to an anonymous reviewer for indicating the need to clarify this point.)

Now, we note that any one knower may, for all practical purposes, regard her own experiential world as being "real for her": She can gain both cognitive and noncognitive knowledge of it through the process of *learning*, which is a most diverse activity of perception and reflection. Thus she can, if she is so inclined, choose to think of the totality of her experiences – or, equivalently, the totality of knowledge that she has constructed at any one time – as constituting "reality for her," and claim that this knowledge tells her something about the real world that she inhabits. But note that this is a *subjective* notion of reality, not an objective reality as envisioned above; it is, in fact, just a synonym for what we have defined as the knower's *experiential world*.

This has consequences for the epistemology of radical constructivism – in particular, its conception of the notions of *truth* and *knowledge sharing*. Radical constructivism does not allow for knowledge to be validly described by such absolute dichotomies as "right/wrong" or "correct/incorrect" or "true/false." (This holds even for mathematical propositions, as will be discussed below.) Of course, any particular piece of knowledge that has been constructed by one individual person may be recorded – say, in written form – and made accessible to other individuals. But they must then in turn construct their own knowledge, from their perception and processing of this recorded information. And this of course raises the issue of whether it is possible for them to share knowledge, in a meaningful sense. According to radical constructivism, knowledge is in essence a private affair: it resides in the mind of some individual knower. So, how can one ensure that two individuals possess the same knowledge about some particular topic – indeed, what does "the same" mean here? The answer is that this is possible for *cognitive* knowledge because this is based on certain rules of argumentation, which can be agreed on by the participants. But it is not possible for knowledge that is *noncognitive*: in this case, there are no such rules that can be agreed on!

Let us consider some examples: I can show someone (assuming she is interested) how to solve quadratic equations; this knowledge is derived from mutually agreed rules of mathematical deduction and logical inference. Similarly, I can demonstrate to her how Newton's law of universal gravitation operates and how it can explain many observable physical phenomena in the Solar system; this is done by using the same rules of deduction and inference, together with physical data obtained through mutually agreed procedures of observation and measurement. In both these cases (taken from mathematics and natural science), the resulting knowledge is *cognitive*: it derives from rules that we both agree on, and it can therefore be regarded as shared between us. But in contrast to this, consider the following instances of personal knowledge, as possessed by some individual knower X: she likes

1| The famous first sentence of Wittgenstein's major philosophical treatise, *Tractatus Logico-Philosophicus*, first published in German in 1921, reads: "The world is all that is the case."

2| By this we mean: a mode of reasoning that does not rely on personal preferences, likes/dislikes, or beliefs. The OED defines cognition as "… knowing, perceiving, or conceiving, as an act or faculty distinct from emotion and volition."

Beethoven's piano concertos; she prefers coffee to tea; she is fascinated by the intuitionist approach in mathematics; she believes in God. All these are examples of *noncognitive* knowledge, which cannot be communicated. Thus X can tell me that she believes in God, but she cannot communicate to me the quality of this belief (how it "feels" for her to be a believer), nor can she demonstrate to me why I too should believe, using rules that we both can agree on. So, the upshot of all this is: cognitive knowledge can be shared, noncognitive knowledge cannot.

Now, we take a look at the opposing epistemic positions of *realism* and *relativism*. For our present purposes, these may be described in simple terms as follows:

Realism

asserts that there exists an objective reality, which is independent of human observers, *and* that it is possible through rational reasoning to attain true (correct) knowledge of this reality. In other words, it is in principle possible to discover an objectively true representation of at least some part of it – for instance, in the context of natural science it can open up the possibility of finding the Final Theory of physics (see, e.g., Weinberg 1993).

This conception of the notion of truth is often referred to in the literature as *truth by correspondence*: a proposition is true if and only if it corresponds to (i.e., gives a correct description of) the real world – in a logical formulation: the proposition p is true, iff p. This is often supplemented by the assumption that true knowledge must be based on a criterion of "justified true belief": it is not enough that someone claims to know something – there must also be some demonstrable *justification* for believing that this knowledge is in fact true.

Relativism

asserts that it is not rationally meaningful to speak of such an objective reality. *Any* piece of knowledge is (and must be) constructed by some individual person, for some specific purpose, and in some particular context; and its "truth-value" can then only be determined relative to this purpose and context. Moreover, the choice of this purpose and context is then solely up to the constructing individual; there is no predetermined objectively correct way of doing it. (This is essentially a paraphrasing of the defining proposition RC2 of radical constructivism.)

This conception of truth is often referred to as *truth-by-context*. It states that a proposition cannot be legitimately said to be true *in itself* – that is to say, in an objective sense – but only true relative to some given *context*: a conceptual scheme, a social group or practice, a person, a religion, an ethical code, etc. In mathematics, this conceptual scheme would then consist of a set of axioms – we will return to this below.

Clearly, radical constructivism offers a relativist epistemology. And note that "offers" is the operative word here. Radical constructivism does *not* claim that its epistemology is the objectively "true way" to address issues of knowledge: its generation, scope, and validation. Indeed, such an assertion would contradict and undermine the theory's basic tenet that the truth-value of any proposition is relative! It would be more accurate to say that radical constructivism *offers itself* as an epistemic approach to any individual knower who may find that this theory resonates well with her own way of thinking about such matters. And, if she should then decide to accept this offer (i.e., adopt the epistemology of radical constructivism), this would be a choice of personal *ontology* on her part – she would embrace this theory as describing her world in a way that fits in well with her personal perception and comprehension of it. But again, this will be part of her store of *noncognitive* knowledge; and hence she will not be able to share it with another person, in the sense defined above. She can, of course, *tell* the other how she experiences this knowledge, assuming that they both speak the same language. But this is a matter of personal belief or preference on her part, and she has no means of demonstrating to the other that she should believe or prefer it too.

2. Mathematics

We now ask: how do the various disciplines of *mathematics* fit into the epistemic framework of radical constructivism? The short answer is that they fit quite nicely: a mathematical theory is indeed a prime example of a theoretical model, constructed by its practitioners (mathematicians). The model may then be designed to describe and explore, in mathematical terms, some part of the natural world; this would then identify it as belonging to the realm of *applied mathematics* (for instance, theoretical physics). On the other hand, it need not be associated with any practical application. Its purpose may be to investigate the logical implications of some particular abstract theoretical structure, without relating the results to any particular physical phenomena; this kind of activity is generally thought of as *pure mathematics*.

At a second glance, however, the situation is not quite that simple. Questions arise, such as: Are there *objectively true* mathematical propositions – i.e., propositions that are "true in themselves," independently of any chosen theoretical model? If so, how would this constrict the permissible choice of mathematical model – surely, a theory from which one could deduce a contradiction to a true proposition cannot be trusted? Where is mathematics situated in the division between realism and relativism, as this is described above?

To illuminate such issues, it is instructive to look at how mathematics developed. The following is just a brief (and necessarily simplified) historical run-through, to illustrate the connections with constructivism.

2.1 The axiomatic approach

The origin of mathematical thinking would seem, as far as can be known today, to have its roots in concrete practical needs. As human civilisation developed in early times, it became necessary to handle a quantitative treatment of various items. Thus, for instance, we may consider the acts of counting and subdivision (for instance, adding-up and weighing of goods, for trade), leading on to the conception of integers and fractions, which form the basis of *arithmetic*, the art of using numbers and computations. Other such deliberations addressed the topic of *geometry* – dealing with notions of position, angle, length, area, and volume – which was clearly also of considerable commercial interest (e.g., for the buying and selling of plots of land). Gradually a set of concepts and computational procedures emerged and developed. It should be noted that this "proto-mathematics" was at the start a very practical enterprise: a set of concrete procedures serving the needs of trade, production, and administration.

As experience with the concepts of mathematics accumulated, many rules and regularities emerged. It was, for instance, noticed early on that the ratio of the circumference of a circle to its diameter seemed always to have the same value (a number slightly larger than three, conventionally denoted today by the symbol π), and that the interior angle sum of a triangle appeared always to be the same (a value conventionally set to 180°). And the contemplation of such rules and regularities then led to a daring and remarkable leap of the imagination: the idea that it should be possible to summarise them in a *logical system*, where all valid mathematical results (theorems) are logically deducible from a small set of fundamental assumptions, called *axioms*.

The origin of this new "theoretical" way of thinking about mathematics is usually ascribed to classical Greece, though similar ideas also appeared in other societies in early history. However, it is the Greek tradition that has been the main inspiration for the development of science and mathematics, from ancient times up to the present day. So, let us take a look at how they thought about these matters.

First, the basic concepts were sharpened and idealised. Thus, for instance, geometry was regarded as a set of relations between different types of abstract geometrical objects: a *point* had no extension (serving only as a marker of position), a *straight line* had no thickness (serving only to mark the shortest way between any two points on it), etc. Similarly, *numbers* were thought of as quantities that had exact values. Such entities (i.e., objects and quantities) were assumed to actually *exist*, i.e., to inhabit an abstract world of pure ideas, which constituted the basic reality of existence; these entities were often referred to as *ideal forms*. The task of mathematics, according to this way of thinking, was to establish relations between such ideal forms. This notion of an abstract *ideal reality*, existing somewhere "above" or "beyond" our physical world, dates back to classical Greece, to the philosopher Plato; and the embodiment of it in an epistemological and ontological theory is often designated *platonism*.The physical world that we can observe around us was assumed to form only an *imperfect representation* of the ideal reality. Thus, any points that we might try to draw or mark in a concrete situation must necessarily possess some extension, any straight line that we can draw between points must necessarily have some thickness and can never be "perfectly straight," and an actual concrete measurement (say, of the distance between two points) can never be performed with perfect accuracy, and hence cannot yield an exact value. Nevertheless, these "fuzzy" and inexactly defined physical objects and quantities were taken to reflect in some way the corresponding entities of the abstract ideal world; and moreover, relations between these entities (ideal forms) in the abstract world were assumed to *govern* relations between the corresponding physical entities. In other words, logical investigations of the abstract ideal world – i.e., mathematics – could enable us to draw valid conclusions about the physical world that we inhabit!

We may well focus on this last sentence: it describes a truly remarkable conceptual leap in the way we, as human beings, can deal with the world of our experience. In fact, this approach has been instrumental in the development of modern science and technology. Let us outline just briefly how the abstract ideal world of mathematical entities was organised conceptually:

The basic idea was to establish a certain set of fundamental relations (called *axioms*) connecting the entities of the ideal world, and then to deduce other relations (called *theorems*) to follow logically from these axioms. The goal was to define a basic set of axioms that is *minimal* (no single axiom should be deducible from the other axioms of the set), *complete* (all valid theorems should be deducible from the chosen set of axioms), and *consistent* (it should not be possible to deduce logical contradictions from them). It may be noted that the number of axioms in the set is generally small; and the beauty of mathematical theory is then that so many interesting theorems can be deduced from such a slender axiomatic base.

It should be noted that this idea of an ideal reality (often called the *platonic world*) that governs phenomena in the observable physical world places mathematics firmly on the *realist* side in the epistemic divide. It is then possible to discover how physical phenomena behave by reasoning about entities and relationships in the abstract platonic reality. Hence there exist objectively true propositions about the world – namely those that reflect relations that hold true in the ideal world. This hegemony of epistemic realism in mathematics (and in natural science) reigned more or less supreme in European philosophy up until the last two centuries, when it began to give way. We may illustrate this process by briefly examining two branches of mathematics: geometry and number theory.

2.2 Geometry

In the tradition of classical Greek philosophy, several attempts were made to collect the various known rules of geometry, and present them within the theoretical framework of an axiomatic system. The most important was that proposed by Euclid, around 300 BC – a theory that has become known as *Euclidean geometry*. A modern formulation of this theory now features a minimal and consistent set of explicitly stated axioms.[3]

In Euclid's original formulation, the theory featured a number of *definitions* of objects such as points and straight lines. The definitions were often rather vague and intuitive: for instance, a straight line was said to be "a line that lies evenly with the points on itself" – hardly a helpful description! Moreover, Euclid considered the fundamental propositions of the theory (from which all theorems are to be derived) to be of two kinds: *axioms* (self-evident truths), and *postulates* (true propositions that are not self-evident but nevertheless cannot be deduced from the other fundamental propositions of the theory). This was in accord with the prevailing philosophy of the time: geometrical objects and relations were assumed to actually *exist* in an abstract ideal

3| As it turns out, it is not always possible to obtain a set of axioms that is *complete*. In 1931 the logician Kurt Gödel showed that any axiomatic theory above a certain (quite modest) level of complexity must necessarily contain theorems that cannot be deduced from the axioms – a result known as Gödel's Incompleteness Theorem.

world of their own, accessible to us through logical reasoning. Hence, it made sense to look for the "correct" definition describing these objects, and the "true" relations (whether self-evident or not) that obtain between them.

The approach of modern mathematical thinking is different – mainly because it does not explicitly invoke any "ideal reality" of existing objects. Thus, the notion of "self-evident truths" (that hold true in this abstract world) is not relevant, and hence Euclid's distinction between axioms and postulates is no longer upheld. An *axiomatic theory*, as it is conceived of today, is formulated as a set of fundamental propositions (which are all referred to as axioms); and from this fundamental set, theorems are to be deduced. Moreover, the definitions of geometrical objects are not assumed to be independent of the axioms: rather, the axioms serve to define these objects! This difference in viewpoint can be illustrated as follows: in the platonic world, there exist certain entities called "straight lines," and these then turn out to obey the Euclidean axioms; in modern mathematical philosophy, the axioms together define a kind of specific entities that have certain properties, and these entities are then given the name "straight lines."[4]

Now, as it turned out, there was one particular axiom (actually a postulate, in Euclid's original formulation) that attracted special attention from mathematicians almost from the beginning. This was the so-called *Euclidean postulate of parallels*, which can be stated as follows:

If a plane α contains a straight line *l* and a point *P* not on *l*, then it is possible to draw *one and only one* straight line *l′* that lies in the plane α and passes through the point *P*, such that *l′* does not intersect *l*. The lines *l′* and *l* are then said to be *parallel* to each other.

It was suspected early on, not that this proposition was incorrect, but that it was not an *independent* postulate – on the contrary, that it should be deducible as a theorem from the other axioms and postulates of the theory! Through the centuries following Euclid, much effort was expended in trying to prove that this was true. The work culminated in the first half of the 19th century, when two mathematicians – Janos Bolyai and Nikolai Lobachevsky – showed, independently of each other, that: (a) the Euclidean parallel postulate is indeed independent; and (b) it is possible to replace it by another fundamental assumption about parallel lines – thus producing a geometrical theory that is different from Euclidean geometry but equally consistent! (This non-Euclidean theory was later, for reasons that need not concern us here, given the name *hyperbolic geometry.*) The new (non-Euclidean) postulate of parallels states:

If a plane α contains a straight line *l* and a point *P* not on *l*, then it is possible to draw *an infinite number* of straight lines *l′*, *l″*, …, lying in the plane α and passing through *P*, that are all parallel to *l* (in the sense that none of them intersect *l*).

Here one might well object that this cannot be true: if we extend the lines *l′*, *l″*, …, most of them will surely intersect the line *l* somewhere in the plane! In fact, one might feel intuitively that the Euclidean postulate "has to be right": since the lines will come steadily closer to each other when both are extended, it seems obvious that they must eventually intersect. But this argument misses the mark: the point is that the theory proposed by Bolyai and Lobachevsky is every bit as *logically consistent* as Euclidean geometry. In other words, straight lines *in the ideal reality* may behave in the Euclidean way, or in the non-Euclidean way – and we have no means of deciding logically between these two possibilities! One example, to illustrate the difference between them: in Euclidean geometry, the sum of the interior angles of a triangle is (as is well-known) always equal to 180°; in hyperbolic geometry this sum is always *less* than 180°, and this sum decreases when the size of the triangle increases!

The appearance of this non-Euclidean geometric theory had a truly momentous impact on the epistemology of mathematics. Recall that the traditional view of geometry was that it described *factual relations* that hold true in the abstract world of ideas – and therefore in turn determine uniquely corresponding relations in the physical world of our experience. Since by assumption there was only one ideal world, and only one physical world, it would seem to follow that there can be only one "true geometry" – i.e., one uniquely defined geometric theory that gives the correct description of the ideal (and hence also of the physical) world. Up till less than two centuries ago, this true theory was naturally assumed to be *Euclidean* geometry, since there were no other candidates at hand. But now a rival theory had appeared, and thorny questions arose such as: How can two different geometries be valid in the same ideal world? And which one of them, if any, describes correctly the physical world? Can it be that different parts of the physical world obey different geometries? Indeed, the very connection between the geometry of ideal reality (as obtained by logical reasoning) and the geometry of physical space (as obtained by observation and measurement) was put into question!

As it turned out, the appearance of hyperbolic geometry was only the start. Very soon after, another – and, from a logical point of view, equally consistent – non-Euclidean theory (called *elliptic geometry*) was proposed. This geometric theory featured a new "parallel postulate," saying in effect that there are *no parallel lines*: any two straight lines that lie in the same plane will *always* intersect! (To illustrate the difference: in elliptic geometry the sum of the interior angles of a triangle is always *greater* than 180°, and this sum increases when the size of the triangle increases!)

In the years that followed, a large number of geometric theories were formulated and studied. It is now generally recognised that there is a great deal of freedom to define different sets of axioms, thus creating different geometries, all of which have equal logical validity. Hence the notion of a unique ideal world, featuring true geometric relations that are realised (although imperfectly) in the physical world, was no longer tenable. Rather, the prevailing attitude now

4| Note that this describes the *epistemology* of modern mathematics – a philosophical approach that is relativist, in the sense that any proposition/theorem requires a formal proof, based on some accepted set of axioms and rules of logical inference, and that there is a wide range of different ways to choose this set. However, it may be conjectured that the *ontology* of most mathematicians is, in fact, one of realism – specifically, some variant of *Platonism*: a belief/conviction that mathematical reasoning in some deep way reflects a platonic world of ideal forms.

is that scientists (in particular, physicists) should look for the geometrical theory that fits best with whatever physical phenomena is being examined.

2.3 Number theory

The conception of numbers arose out of very practical considerations (the counting and measurement of concrete physical objects), as was remarked above. Gradually, however, a *theory* of these entities (an arithmetic) developed, where properties of numbers and relations between them were to be derived from certain fundamental propositions (axioms). The definition of the basic entities then went through a well-known process of extension: starting with *integers* (whole numbers), one extended the set of axioms to define *rational numbers* (fractions) – which include integers as a special case (fractions with unit denominator). From there, the axiomatic basis was further extended to define *real numbers* (comprising rational and irrational numbers), *complex numbers* (comprising real and imaginary numbers), etc.

Note that this progressive extension of the concept of a number rapidly loses direct contact with its concrete practical origins. Let us consider the operation of a concrete *measurement* of some quantity associated with a given physical object. This is, in principle, done by comparing the object with some conventionally chosen unit; and the result of the comparison is then given by a *rational number* a/b, where a and b are non-zero integers. Thus, for instance, the statement that a given length l is $5/8$ meters long is equivalent to saying: if a meter stick is divided into eight equal segments, and the length l is laid along the stick, it will cover five of these segments. So, the upshot of this is: *physical measurements yield rational values*. Moreover, measurements can in principle be performed to any desired accuracy by suitably increasing the numerator and denominator of the rational measurement value. (Thus, for instance, the statement $l = 3557/10000\,\text{m}$ means that the length l is given as $35.57 \pm 0.005\,\text{cm}$.) Note that this procedure can never yield an *exact* value: there will always be some inaccuracy in any measurement! This relates to the fact that the set of rational numbers is infinite and *dense*: for any two rational numbers x and $y > x$, it is always possible to find a rational number z lying between x and y, i.e., such that $x < z < y$.

Early on, it was realised that there are numbers that cannot be expressed as fractions: one example is $\sqrt{2} = 1.4142\ldots$ Such non-fractional quantities, whose values are generally defined by converging infinite sequences, were given the somewhat unfortunate appellation *irrational numbers*. The set of irrationals is, like the set of rationals, infinite and dense; and moreover, the two sets are *densely interspersed*, in the sense that it is possible to find rational numbers with values between any two given irrational numbers, and vice versa. Together, the rational and irrational numbers constitute the set of *real numbers*.

It is the real numbers that form the basis of mathematical analysis, as applied to science. But note that it is only the rational numbers that are conceptually connected with physical measurements: one never actually *measures* (in the concrete sense of comparing object and unit) an irrational value for a given physical quantity! Thus, we may pose the question: Why does the formal manipulation of scientific quantities require that one uses the complete set of real numbers? Would it not be more logical (and economical) to stay with the rational numbers – which, after all, are the ones directly associated with the act of physical measurement – and base the mathematical analysis of scientific laws on them?

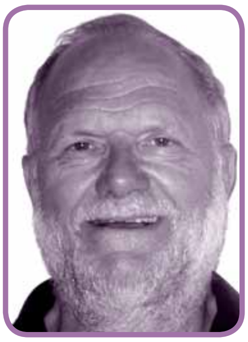
The answer is that a theory of mathematical analysis that admits only rational numbers, though possible in principle, would be quite intractable in practice! The reason is, somewhat loosely speaking, that the set of rational numbers by itself is "not dense enough." In this analysis, one considers algebraic equations of a very general kind; and it would be extremely cumbersome (to say the least) to have to "filter out" all equations that have irrational solutions, leaving only equations with rational solutions for the legitimate description of scientific relationships. (As a very simple example, one would then have to allow the equation $x^2 - 4 = 0$, which has the rational solutions $x = \pm 2$, but disallow the equation $x^2 - 2 = 0$, with the irrational solutions $x = \pm\sqrt{2}$.) In fact, since rational and irrational numbers are densely interspersed, this would play havoc with the whole notion of *continuity*, which is fundamental in the treatment of mathematical functions.

To avoid such formal complications, we allow both rational and irrational values for physical quantities as solutions emerging from the mathematical treatment of scientific laws governing these quantities. In other words, we accept the physical validity of propositions such as "the length l of this object is $\sqrt{2}\,\text{m}$," and take it to have the following operational meaning: when the length l is actually measured, the result (in meters) will be a rational number that, with an increasing accuracy of measurement, can be assumed to come arbitrarily close to the "exact value" $1.4142\ldots$ of the irrational number $\sqrt{2}$, as determined by mathematical theory. But note that it will never actually "get there"; no matter how much we increase the accuracy, there will always be an infinite number of rational (and irrational!) values between the measured value of l and $\sqrt{2}$.

Thus, as we can see, the definition of a number has been extended from its "physical origins" (as a result of counting and measurement), to allow for a more convenient mathematical treatment of scientific relationships. And note that this process of extension does not stop here. Many algebraic equations have solutions that are not real numbers – one simple example is $x^2 + 1 = 0$. The desire to include such equations then leads to a further extension of the axiomatic set, to define the so-called *complex numbers*, which admit values involving square roots of negative real numbers. (Such values have been given the rather unfortunate name *imaginary numbers*.) And indeed, this extension has also proved to be very useful in the mathematical treatment of science.

2.4 The game of mathematics

The brief discussion presented above, addressing some examples from geometry and arithmetic, demonstrates how mathematical concepts and procedures, though originally inspired by very practical concerns (counting and measuring of physical objects), have rapidly moved away from the "real world" and become part of an intellectual activity in its own right – an activity that may manifest itself either as *applied mathematics* (a modelling of some particular physical system) or as *pure mathematics*



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(theoretical deliberations that claim no connection with any part of the physical world). In fact, the distinction between these two areas of mathematics is not always clear-cut; numerous links may easily be found. Thus, for instance, the conception of number – originally a tool for practical needs – has developed into many and various abstruse specialities, such as the study of prime numbers, which have no obvious connections with the physical world. (Thus consider, for example, the famous *Goldbach conjecture*, which states that any even number can be expressed as a sum of two primes. Despite a lot of effort, no one has up to now been able to either prove or disprove this seemingly simple proposition!) And on the other hand, purely theoretical constructs, which were initially proposed on an abstract level, have often proved subsequently to have applications in physics. Thus, for instance, four-dimensional *Riemannian geometry* has turned out to be essential as a basis for Einstein's theory of general relativity, which is presently the accepted physical theory of gravitational interactions. Here one may also cite the example of *string theory*: the attempts by physicists to describe elementary particles in terms of small one-dimensional entities (strings) that move and interact in a space of at least ten dimensions!

The scenario that emerges from all this is that the epistemology of mathematics has moved away from an initially *realist* position: i.e., featuring a set of objectively true propositions describing properties of the ideal platonic world, and hence by extension also properties of the physical world. The present epistemic framework is one of *relativism*, where the truth-value of any mathematical proposition is only defined relative to some particular context. This is the viewpoint advocated by radical constructivism: any mathematical knowledge – whether pure or applied – is (and must be) *constructed* in the mind of the knower to model (i.e., describe and explore) some part of her experiential world. Thus there is no such thing as "the right mathematics," lying out there waiting to be discovered; in other words, there exists no objectively true mathematical theory to describe the world that we human beings can experience. When we want to address some particular topic, we are free to construct a mathematical model of it by choosing an axiomatic base and exploring the consequences of this choice; and this can be done in many different ways.

This is certainly the case in *applied mathematics*, as has been argued above. But it also applies to *pure mathematics*. For a "pure mathematician," the discipline is well worth pursuing for its own sake – for her own personal enjoyment, so to speak; certainly she feels no need to justify this activity in her mind by requiring that it should in some way describe how the physical world works. Indeed, it seems more appropriate to say that she engages in mathematics as an *intellectual game*! Let us elaborate a little on this imagery.

In the "game of mathematics," the players (mathematicians) adopt certain *game rules*, defining a set of axioms and principles of admissible logical inference; and they are then free to explore the various consequences that follow from these rules. Note that there is a large amount of freedom in the choice of rules; in general, they must conform only to requirements of consistency (i.e., not lead to logical contradictions). Thus there are no uniquely determined "correct rules" waiting to be discovered – any set of rules that leads to a consistent theory is in principle permissible as a valid object for study. Hence the classical idea of an *a priori* defined "factually true" set of mathematical rules, residing in an ideal abstract world and governing observable phenomena in the physical world, is no longer tenable – as has already been argued above. We may formulate any number of mathematical theories that are "true," in the sense that they conform to the selected rules of the game; and these rules are then conventional, with a large freedom of choice. The fascination of this game is that it is so "rich": i.e., that so many very elaborate and (to mathematicians, at least) interesting theories can be deduced from fairly simple rules. To sum up, mathematics is not something we can "find in nature"; it is a theoretical *construction*, made in such a way as to satisfy the rules chosen by its creators (the mathematicians).

As an analogy, we may consider the game of *chess*. It is defined by a simple set of game rules, describing the arena of play (the chess board) and the permitted movements and interactions of a given set of agents (the chessmen). The deployment of these rules then yields, as is well known, a game of remarkable variation and richness: a chess match can evolve in a literally astronomical number of different ways; and the planning of just a few moves ahead, taking account of all possible strategies that can be devised by your opponent and your own response to them, rapidly becomes prohibitively complicated. But note that there is nothing inherently unique or mandatory about the *rules* of chess! Historically, these rules were proposed some centuries ago; and today they are agreed, by universal convention among chess players, to define "the way to play the game." Nevertheless, it is a notable fact that many chess players have experimented with alternative rules: expanding the number of squares on the board, and/or introducing new kinds of chessmen with new rules for moving and interaction (so-called "exotic chess"); stacking chess boards on top of each other, with rules determining how the chessmen can jump between boards ("three-dimensional chess"); conceptually connecting the side edges of the board, enabling chessmen to move directly from the left-hand edge of the board to the right-hand edge ("cylindrical chess"), etc. However, these alternative chess games, though claimed by the aficionados to be great fun, have not acquired a large following; in the mainstream, classical chess still rules!

Of course, this conception of mathematics as an intellectual game played by the mathematicians for their own enjoyment is too simplistic. It does indeed describe the attitude that many of the practitioners of mathematics have towards their subject; but if that were the whole story, it would indeed be difficult to defend the strong position that this discipline holds in school education. We also need to take into account its *usefulness*: it is a fact that much mathematics has been developed for the explicit purpose of application to science – and that this approach has often worked extremely well! The physicist Eugene Wigner once published a paper named "On the unreasonable effectiveness of mathematics in describing the physical world" (1960), remarking on this point. On the other hand, others[5] have claimed that this is not so unreasonable: much mathematical theory is, in effect, designed to describe aspects of the physical world; and so one may be gratified, but perhaps not overly surprised, to discover that it does the job well! Be this as it may, it is a fact that many mathematical theories, originally proposed and developed as "pure mathematics" without any thought of application in science, have subsequently been found to be very useful in describing aspects of the physical world. It is clear that, while mathematics may not need science, science definitely does need mathematics!

5| Notably Andrew Pickering (1986), also a physicist.

Let us try to push this game metaphor a little further. Recall that in radical constructivism, knowledge is taken to be constructed, to serve as a *model* of some part of the experiential world of the knower. Clearly, the metaphor fits well in the context of applied mathematics: the game is then defined as the act of modelling (i.e., describing and exploring) some definite set of observed natural phenomena. And in the case of chess, of course, the subject of this modelling is obvious: a military battle, as this was conceived of in ancient times, with two armies (black and white chessmen) marched up against each other, featuring infantry (the pawns), cavalry (the knights), mobile assault towers (the rooks), etc. The game is then played, using the agreed rules of chess, with the goal to conquer (i.e., checkmate) the opponent player.

However, in the context of pure mathematics, one may ask: what "experiential world" of the knower is being modelled in this case – what is the game, and why is it being played? The following answer may be suggested: the pure mathematician is engaged in playing with some selected set of mathematical constructs and axioms, which forms the *subject* of the model, using rules of logical inference. These items (constructs, axioms, and rules) then form that part of her experiential world that she wants to model. The goal of the game is to explore the consequences of her choice of subject, i.e., to deduce theorems describing various properties of the model. And she is playing this game for her own personal intellectual enjoyment, as already noted, and quite often will not be expecting any results of practical value to come out of it.

3. Conclusion

According to radical constructivism, mathematics does not (and cannot) give us a (potentially or actually) *correct* knowledge of the world. On the contrary, it presents itself as a technique for modelling the experiential world of individual knowers; using this technique, we may generate knowledge through a process of individual *construction* in the mind of the knower. As is well-known, this technique has been extremely successful in expanding our understanding and mastery of the world we live in.

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