Infinity and the Observer: Radical Constructivism and the Foundations of Mathematics

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> **Problem** • There is currently a great deal of mysticism, uncritical hype, and blind adulation of imaginary mathematical and physical entities in popular culture. We seek to explore what a radical constructivist perspective on mathematical entities might entail, and to draw out the implications of this perspective for how we think about the nature of mathematical entities. > **Method** • Conceptual analysis. > **Results** • If we want to avoid the introduction of entities that are ill-defined and inaccessible to verification, then formal systems need to avoid introduction of potential and actual infinities. If decidability and consistency are desired, keep formal systems finite. Infinity is a useful heuristic concept, but has no place in proof theory. > **Implications** • We attempt to debunk many of the mysticisms and uncritical adulations of Gödelian arguments and to ground mathematical foundations in intersubjectively verifiable operations of limited observers. We hope that these insights will be useful to anyone trying to make sense of claims about the nature of formal systems. If we return to the notion of formal systems as concrete, finite systems, then we can be clear about the nature of computations that can be physically realized. In practical terms, the answer is not to proscribe notions of the infinite, but to recognize that these concepts have a different status with respect to their verifiability. We need to demarcate clearly the realm of free creation and imagination, where platonic entities are useful heuristic devices, and the realm of verification, testing, and proof, where infinities introduce ill-defined entities that create ambiguities and undecidable, ill-posed sets of propositions. > **Constructivist content** • The paper attempts to extend the scope of radical constructivist perspective to mathematical systems, and to discuss the relationships between radical constructivism and other allied, yet distinct perspectives in the debate over the foundations of mathematics, such as psychological constructivism and mathematical constructivism. > **Key words** • **Foundations of mathematics, verificationism, finitism, Platonism, pragmatism, Gödel's Proof, Halting Problem, undecidability, consistency, computability, actualism.**

… it is unintelligible to attribute existence to anything that cannot or could not at some time be perceived. (Ernst von Glasersfeld 2007: 97)

For my part I think, and I am not alone in so thinking, that the important thing is never to introduce any entities but such as can be completely defined in a finite number of words. (Henri Poincaré 1952: 45)

Again, nothing infinite can exist; and if it could, at least the notion of infinity is not infinite, i.e., does not contain an infinite number of marks (Aristotle, Metaphysics, 994b27-28, in McKeon 1941)

Evidently, the category of number is wonderfully consistent and complete as long as it is applied to counting real apples, but it becomes paradoxical when it is extended to such things as infinite sets, which transcend our experience. (Gunter Stent in Delbrück & Stent 1986: 11)

Introduction

In this paper we attempt to draw out how a radical constructivist epistemology might handle mathematical ideas, formal systems, and infinities. Our motivation arises from a desire to clarify contemporary thinking about the nature of mathematical objects, formal operations, and proof. We present an overview of the different perspectives in the debate about the foundations of mathematics and discuss compatibilities and incompatibilities with radical constructivism. In turn we discuss mathematical psychologism (intuitionism, constructivism), mathematical realism (Platonism), formalism, and finitism. The foundations crisis concerned the inability to prove the consistency of arithmetic on the natural numbers, but we argue that this is not at all a problem for the varieties of physically-realizable, finite mathematics we use for our computations.

Arguably, the crisis of the foundations of mathematics that erupted in the early 20th century was precipitated by introductions of self-referential meta-mathematical statements and Cantorian infinities into proof theory, and there is no crisis provided that formal systems are instead kept strictly finite. In this respect, the "loss of certainty" in the logical consistency of mathematics paralleled the crisis in the foundations of physics, which shook realists' faith in absolute, observer-independent physical reality. Because neither belief in physical reality nor in mathematical consistency has any concrete practical implications, neither crisis impeded the advance of science and mathematics.

In order to explore what the foundations of mathematics might mean for radical constructivism and vice-versa, a rough map of the philosophical territory is a useful place to begin. Although radical constructivist writings have addressed the development, use, and elaboration of mathematical concepts (Glasersfeld 1991, 2006; Piaget 1980), relatively little has yet been written about mathematical foundations from this perspective (e.g., Stolzenberg 1984). It therefore appears that we are treading mostly on new ground. Perhaps this is not so surprising, given the highly abstract and ontological orientation of much of the philosophy of mathematics and the observer-grounded, epistemological stance of radical constructivism.

Radical constructivism provides ontological and epistemological frameworks, as well as theories of observation, action, communication, knowledge, thought, and learning. We believe that radical constructivism as a broader philosophical movement also encompasses general theories of observer-actors, of adaptive cybernetic percept-action systems, and of self-constructing neural systems. Radical constructivism, coming from an epistemology of situated, self-constructing observers, is naturally allied with different psychological, pragmatist, constructivist and finitist perspectives in the philosophy of mathematics. Although we want to avoid simplistic conflations of radical constructivism with these different schools of thought, neither do we want to understate what they have in common.

The different facets of radical constructivist theory have relevance for different aspects of mathematics. For example, psychological approaches to mathematics involve understanding how mathematical ideas are constructed in our minds. Mathematical constructivism, on the other hand, is operationalistic, not psychological, and involves concrete methods by which one constructs formal mathematical objects whose properties can then be examined. Finitism involves beliefs about the ontological and epistemological status of potential and actual infinities.

An important guiding question for the radical constructivist project is how far it should go, if at all, to avoid talk of the possible in favor of what is immediately available to experience, i.e., how to weigh up the balance between free imagination and concrete demonstration. Opposed to psychologism, constructivism, and finitism are realist, platonic philosophies of mathematics that hold that mathematical objects have an objective existence independent of the mind and therefore are discovered by us rather than invented. Platonists are consequently willing to accept the existence of entities, such as infinities and hierarchies of infinities, that they can imagine but not perceive. Much of the debate over mathematical foundations concerns differences over what counts as a real or definite mathematical object, how number systems are to be constructed and/or justified, what constitutes mathematical truth, and what logical methods are admissible in mathematical proofs.

Philosophies of mathematics

Historically, the different camps in the foundations debate have been divided into intuitionists and (mathematical) constructivists, formalists, and platonists. Finitists, whom we will discuss later, are varieties of mathematical constructivists.

The intuitionists (L. E. J. Brouwer, Arend Heyting, Hermann Weyl, and perhaps Ludwig Wittgenstein) considered mathematical concepts to be intuitive, mental constructs that require external, concrete constructions in order to be reliably tested. Intuitionists adopted constructivist approaches because they had doubts about the self-evident nature of mathematical truth and the universal validity of classical logics (such as the law of the excluded middle, which asserts that statements are either true or false). If one adopts a constructivist, verificationist concept of truth and meaning, such that statements are only meaningful in relation to how they are interpreted or verified, then there are three truth values: verified true, verified false, and unverifiable. This last category of "unverifiable" is appropriate for ill-posed, self-contradictory, and/or meaningless statements.

The formalists (David Hilbert, John von Neumann) held that mathematical ideas should be expressed in terms of explicit, concrete formal procedures (computations) operating on symbols. Like operationalism in physics, formalism allows metaphysical issues to be sidestepped, if one wishes, by focusing exclusively on the meaningless, but reliable, manipulations of symbols. But unlike constructivism, the central goal of the formalist program was to rationalize logical reasoning about mathematical entities, whatever their ultimate nature, and therefore the formalist program accepted as valid nonconstructive proof methods based on logical assumptions that the constructivists rejected as unjustified (indirect existence proofs of entities that cannot necessarily be constructed, undecidability proofs by contradiction, proofs based on properties of infinite sets).

Finally, the platonists, or mathematical realists (Georg Cantor, Richard Dedekind, Kurt Gödel, Alfred Tarski, Alan Turing) believed in the objective existence of infinite sets, the reliability (or infallibility) of classical logic, absolute mathematical truth, and the usefulness of self-referential statements. Platonic-realist accounts dominate the philosophy of mathematics literature, where representative popular and academic treatments can be easily found (Rucker 1982; Moore 1990; Maddy 1990).

Mathematics and the physical world

What is the status of mathematical ideas from a radical constructivist perspective? On the one hand, mathematical ideas are concepts that are held by individual human beings, and therefore no different qualitatively from any other concepts that are constructed through experience. On the other hand, mathematical ideas, when they have proven consistent through their successful use, give the illusion that, rather being mental constructs, they are in some sense inevitable expressions of the inherent order of the universe. In this respect our naïve intuitions about mathematical reality parallel those concerning physical reality and objective truth that radical constructivists have taken deep pains to dismantle (Glasersfeld 1996).

This appearance of universality and inevitability is reinforced by the consistency and reliability of computations that we rely on in everyday life. Normally, we do not stop to question the consistency of the arithmetic operations that we use to count or exchange money or to interconvert physical measured units and estimate quantities (e.g., time, distance, volume, weight, temperature). Many years ago I came across a *New Yorker* cartoon in which a restaurant patron, examining his check at the end of a meal, says in effect to the waiter, "Although I do not doubt the correctness of your arithmetic on this bill, I have fundamental doubts about the consistency of the foundations on which it is based." What makes the cartoon funny is that common sense tells us that the mathematics is sound, yet we hear the distant echo of experts telling us that the foundations are not settled.

Although there are obvious uncertainties and ambiguities at both microscopic and macroscopic levels of description, it is hard to escape the feeling that the physical universe is fundamentally orderly. There are universal physical laws at work, such that knowledge of them permits accurate and reliable prediction of a huge host of physical processes and events. Physics and mathematics so often prove to be highly effective tools for prediction that it is therefore not without some justification that Eugene Wigner could extol the "unreasonable effectiveness of mathematics" for describing the physical structure of the world. As a result of the successes of mathematics and physics, there exists a deeply held and widespread belief that the world itself is mathematical in structure (Kline 1980). In what amounts to a secular religion, Gödel and Turing are regarded popularly as saints, if not demigods, and the prestige of mathematics as an oracle of absolute truth is rivaled only by that of physics (see Rotman 1993 for an embodied deconstruction). To be fair, some of this excessive exuberance and reverence that we see in the popular press is related to the palpable joy of mathematical thinking. Nonetheless, many mathematicians and physicists believe that mathematical truths reveal deep physical truths, and some go so far to hold that the universe is inherently a gigantic computational process, in the image of deterministic cellular automata.

Against these sentiments that mathematics occupies a special relation to reality, constructivist psychological perspectives in the philosophy of mathematics hold that mathematical ideas are concepts that are held by individual observer-actors that are constructed through experience with the world, much in the same ways that other concepts are created (Glasersfeld 2006; Dehaene 2011; Lakoff & Núñez 2000; Piaget 1980). These free creations of the mind have no necessary a priori truth or relation to the physical world, and, like other non-mathematical concepts, their usefulness in guiding thought and action is only proven through their application. Rather than truths that are discovered, it is those mathematical constructions that effectively model the observed behavior of the material world that we select for incorporation into our physical theories and retention in our textbooks.

We argue that in most creative processes, from structural and functional novelty arising through biological evolution to innovations arising through human learning, there are two phases. There is an expansive phase in which many possible alternatives are formulated, and a contractive phase in which these alternatives are rigorously put to tests of efficacy and reliability. In biological evolution, the expansive phase involves genetic variation, whereas the contractive phase involves natural selection. In intellectual creation, the expansive phase is most often the realm of combinatorics of ideas generated by the unfettered imagination, whereas the contractive phase is the realm of the real-world testing of the ideas generated (Cariani 2012). In mathematics, the expansionary, imaginative realm occurs in the mind of the individual mathematician, whereas the rigorous, contractive realm of proof occurs within a community of fellow mathematicians.

All constructivist psychological theories seek to explain how new mental concepts arise from the interaction between adaptive, neural self-organizing processes in the brain and ongoing sensing and acting (sensorimotor) interactions with an external environment. In locating mathematical objects in the experience of the observer-actor, radical constructivism shares some common ground with intuitionist conceptions of mathematical objects as purely mental constructs. These mental constructs have the same ontological status as other kinds of ideas. There are no obvious qualitative neurological differences that would cause us to think that mathematicians receive some sort of special access to the underlying structure of the world (something akin to divine inspiration). Despite the present primitive state of our understanding of the neural mechanisms of thoughts, eventually these processes will be understood. Thus, even though we presently lack crucial neural details, there appears to be no inherent barrier to explaining the generation of mathematical concepts in terms of a neurally grounded constructivist psychological theory.

Mathematical realism

The ontological status of mathematical concepts is more contentious. From ancient times, some philosophical systems have asserted the primacy of ideal forms over the outward appearances that the material world presents to us. We will refer to this belief in an absolute, observer-independent, objective, ideal realm as Platonism (also called formism, Pepper 1942). Platonism is a realist ontology in that it holds that (1) there is an absolute and objective external reality (of ideal forms), and (2) the structure of this realm is at least partially knowable (through mathematics).

One can compare and contrast platonic realism with physical and theological realisms. In physical realism, the objective, knowable reality is the material realm rather than the realm of ideal forms. In theological realism, this realm involves transcendent beings and/or dimensions not necessarily accessible via empirical observation (but perhaps through personal revelation). These are fundamentally ontological differences that reflect divergent metaphysical assumptions about the ultimate nature of the world, and it is extremely useful if one can recognize these modes of thought when they appear.

All varieties of realism reject an insuperable Kantian split between the realm of appearances and things-in-themselves. Radical constructivism, while it does not solipsistically deny the existence of an external world, does deny objective knowledge (veridical representation) of it. As von Glasersfeld (2007: 97) put it, "Radical constructivism does not deny an ulterior reality... it denies that human rational knowledge can attain a God-made world or produce anything that could rightly be called a representation of it."

In contrast to radical constructivists, realists of all three kinds maintain a faith that the structure of that world is in some deep sense knowable apart from concrete observations and experiences. For theists this faith comes from a belief in God, for physical realists, it comes from a belief in the progressive correctness of physics (or science in general), and for platonists it comes from a belief in the universality of mathematics.

The success of modern science in accounting comprehensively for the behavior of the physical world created a centuries-long crisis in the Western belief in an omnipresent, omniscient, and/or omnipotent God. The crisis in the foundations of physics in the early 20th century, precipitated by special relativity, wave-particle duality, and quantum mechanics, was related to the necessity of including the observer and his/her mode of measurement in descriptions of physical events, i.e., it was related to the breakdown of realist ("classical") assumptions about the observer-independent nature of physical descriptions.

Similarly, the crisis in the foundations of mathematics was precipitated by the realization that consistency could not be proved for arithmetic on the natural numbers. As we will argue, this is only a "crisis" of belief if one believes in the reality and consistency of sets of infinite scope. There is no crisis if there are no such sets in the first place or if these abstract, ideal entities really have nothing to do with practical computations.

Formal systems

As far as the imagination goes, radical constructivism is completely open-minded, and places no limits. Any concept that can be conceived may have some usefulness in some pragmatic context. In Feyerabendian terms, in the realm of the imagination, "anything goes" – even ideas that are incorrect, inconsistent, and fanciful can become useful in generating hypotheses about the world.

However, in the realm of testing ideas, radical constructivism is more conservative and skeptical. Purely mental constructs, such as belief in imaginary objects and beings, can be distinguished from products of our senses (observations, measurements). In its critical mode, radical constructivism rejects systems based on imperceptible entities, and restricts consideration to distinctions directly presented to experience, i.e., those that can be reliably made by observers. This permits common distinctions that can be shared and verified by communities of observers and the formation of intersubjectively shared social conventions and mutual understandings.

In its anti-realist empiricism and pragmatism, radical constructivist epistemology is allied with scientific pragmatism, verificationism, and operationalism (van Fraassen 1980; Murdoch 1987; Bridgman 1936). Propositions have no inherent truth or relation to some absolute external reality, and their "truth" is not absolute or observer-independent, but consists of the means by which they are evaluated (verificationism) or used (pragmatism). The means by which a proposition is to be evaluated, its "truth" determined, is a shared, agreed-upon prior convention that is collectively implemented by a community of observer-actors.

Thus, from this perspective, formal (syntactic) truths are propositions that are confirmed or falsified by means of meaningless, conventional formal operations on signs. For example, evaluation of an arithmetic statement ($2+3=5$) requires symbol recognition coupled with purely sign-type-based syntactic rules that successively rewrite the symbols on two sides of the equation until they become the same. In contrast, empirical, (semantic) truths are propositions concerning states of the world that are evaluated using contingent measurement processes appropriate to the meaning of the statement. For example, evaluation of "it is now raining in Rome" requires some measurement to ascertain whether rain is currently falling in that city.

The two kinds of truths are distinct because their modes of verification, via rule-governed computational procedures or contingent measurements, are fundamentally different kinds of operations. As a result, the functions of these two kinds of statements are qualitatively different. Formal truths tell us about the consequences of our formal rules, whereas empirical truths tell us something about the state of the external world (as registered by our measuring devices). Another way of saying this is that mathematics, by itself, tells us nothing directly about the physical world, and in turn, because formal conventions are arbitrary constructs, empirical observations about the physical world do not in any way constrain our mathematics.

Our formal systems and computers are designed to operate according to their own internal rules, to act completely independently of external inputs (perturbations, measurements, oracles). As theoretical biologists Howard Pattee and Robert Rosen have argued, formal systems are material systems so arranged that their functional states and state-transition behaviors can be effectively described in terms of high-level rules, "non-holonomic constraints," such one need not make any reference to the underlying physical laws in order to completely describe their behaviors in these terms (Pattee 1985, 2001; Pattee & Rączaszek-Leonardi 2012; Rosen 1987, 1994). A digital computer is isomorphic to a finite formal system in its behavior, and a given finite formal system can be physically realized by many different material systems.

Formal systems are the vehicles by which mathematical ideas are externalized and outwardly expressed in explicit, symbolic form. In Hilbert's original conception, formal systems can be regarded as externalized, constructed conventions of inherently meaningless but concrete symbol tokens and formal operations on them that are agreed upon and understood by a community of observer-actors. The operations and results are constructed to be unambiguously interpretable, completely reliable, and intersubjectively verifiable, such that all members of the community applying the same rules on the same symbolic objects all reliably arrive at the same results. Insofar as the community can agree on a set of conventional rules that adequately characterize a particular situation (not always trivial, and sometimes not possible) logical disputes and mathematical problems related to these conventions can be straightforwardly resolved through formal construction and calculation. As Gottfried Leibniz suggested in 1685:

> "The only way to rectify our reasonings is to make them as tangible as those of the Mathematicians, so that we can find our error at a glance, and when there are disputes among persons, we can simply say: Let us calculate, without further ado, in order to see who is right." (*The Art of Discovery*, in Wiener 1951: 51)

Hilbert's formalist program

At the beginning of the 20th century, the philosopher and mathematician Hilbert explicated, refined, and systematized the notion of the formal system as the arena for explicating mathematical ideas in symbolic form so that their consistency could be tested by means of formal procedures on strings of symbols (Zach 2009).

Hilbert's program was based upon the construction of formal processes out of material tokens ("extralogical concrete objects") and physical operations on them. While he believed infinite collections of such tokens were not physically realizable, he did believe that consistent finite representations of infinite entities could be constructed and implemented through finitary operations on finite sets of physical tokens.

"As a further precondition for using logical deduction and carrying out logical operations, something must be given in conception, viz., certain extralogical concrete objects which are intuited as directly experienced prior to all thinking. For logical deduction to be certain, we must be able to see every aspect of these objects, and their properties, differences, sequences, and contiguities must be given, together with the objects themselves, as something which cannot be reduced to something else and which requires no reduction. This is the basic philosophy which I find necessary, not just for mathematics, but for all scientific thinking, understanding, and communicating. The subject matter of mathematics is, in accordance with this theory, the concrete symbols themselves whose structure is immediately clear and recognizable." (Hilbert 1964: 142)

These concrete tokens or primitives would then be combined in various ways to construct more complicated relations between the combinations. Von Neumann, neatly summarized the goals of the movement:

"The leading idea of Hilbert's theory of proof is that, even if the statements of classical mathematics should turn out to be false as to content, nevertheless, classical mathematics involves an internally closed procedure which operates according to fixed rules known to all mathematicians and which consists basically in constructing successively certain combinations of primitive symbols which are considered 'correct' or 'proved.' … We must regard classical mathematics as a combinatorial game played with the primitive symbols, and we must determine in a finitary combinatorial way to which combinations of primitive symbols the construction methods or 'proofs' lead." (von Neumann 1964: 61f.)

The goal of formalist mathematics is to construct a system of operations on tokens that is internally consistent, that is, no combination of allowed operations should lead to a contradiction. Hilbert's conception of formal systems, thus, was operationalist and anti-realist and allied in spirit with interpretations of quantum mechanics of Bohr and von Neumann. This kind of formalist finitary reasoning is generally compatible with radical constructivism in that it demands that the symbols and the operations involved are fully accessible to the observer and also that there is no objective meaning or absolute truth attached to those symbols and operations apart from how they are used.

On the other hand, Hilbert very much wanted the conceptual freedom and intuitive power of the notion of infinity, and had famously stated that "No one shall drive us out of the paradise which Cantor has created for us." He sought to demonstrate, using finitary arguments, the consistency of arithmetic on the natural numbers, and he believed that this was possible. In other words, he wanted to make the use of infinities safe for mathematics. Arithmetic with potential infinities in its core would be proven consistent using Hilbert's formalist, finitary, concrete "material logical deduction" framework.

"We have already seen that the infinite is nowhere to be found in reality, no matter what experiences, observations, and knowledge are appealed to. Can thought about things be so much different from things? In short, can thought be so far removed from reality? Rather is it not clear that, when we think that we have encountered the infinite in some real sense, we have merely been seduced into thinking so by the fact that we often encounter extremely large and extremely small dimensions in reality? Does material logical deduction somehow deceive us or leave us in the lurch when we apply it to real things or events? No! Material logical deduction is indispensable. It deceives us only when we form arbitrary abstract definitions, especially those which involve infinitely many objects. In such cases we have illegitimately used material logical deduction, i.e., we have not paid sufficient attention to the preconditions necessary for its valid use." (Hilbert 1964: 142)

However, the structure of this evaluative formalist framework turned out to be inconsistent with his desire to validate the intuitive idea of potentially-infinite number systems by demonstrating their consistency through non-constructive existence proofs. Hilbert disagreed with the constructivist intuitionists over the use of the principle of the excluded middle, i.e., he maintained that a formal proposition is inevitably either true or false.

Radical constructivist thinking about mathematical foundations might likely depart from Hilbert's program on two grounds: because of its end goal of justifying and rationalizing infinitistic entities and because of its abandonment of the construction of mathematical objects.

Gödel's proof

In 1900, Hilbert outlined 23 unsolved problems in mathematics. As the second one, Hilbert posed the problem of proving, through finitary means, that an arithmetic operation on the natural numbers is consistent, i.e., that it does not generate contradictory results. The decades that followed were filled with constructivist-platonist debates over Cantor's taxonomy of infinities, paradoxes raised by self-referential, impredicative statements ("this sentence is false"), the ontological status and meaning of mathematical propositions and proofs, the validity of proofs based on the excluded middle (every proposition is either true or false) and by contradiction (a formal system is inconsistent if a proposition and its negation are both generated), and the construction of sets (what kinds of sets are permitted, such as null sets and sets that contain themselves). Depending on one's degree of trust in the concept of infinity, one accepts into the proof process sets consisting of actual infinities (Cantor, Gödel), potential infinities (Hilbert, varieties of constructivism that assume natural numbers), or only finite, countable numbers of distinguishable objects (Goodman 1956).

In 1931, Gödel proposed two theorems of logic: his first and second incompleteness theorems (Gödel 1931; Nagel, Newman & Gödel 1958). The details of the proofs and subsequent discussions and elaborations form a vast literature. The first incompleteness theorem was interpreted to mean that a system such as arithmetic on the natural numbers is incomplete because there are always true statements about the system that cannot be expressed from within it, undermining Hilbert's goal of showing that all true statements about arithmetic on natural numbers could be expressed in that system. The second incompleteness theorem is taken to mean that a system such as arithmetic on the natural numbers cannot be used to prove its own consistency.

To most observers at the time, including Hilbert himself, Gödel's incompleteness theorems demolished the Hilbertian program of successfully proving the consistency of arithmetic (Nagel, Newman & Gödel 1958; Dawson Jr. 1988). Despite the great weight of mathematical and philosophical opinion behind it, there were always those who strongly doubted the significance and applicability of Gödel's approach (Shanker 1987, 1988), and also whether it necessarily doomed Hilbert's Program (Detlefsen 1979). Three decades after Gödel's proof, Bertrand Russell had doubts: "If you can spare the time, I should like to know, roughly, how, in your opinion, ordinary mathematics – or, indeed, any deductive system – is affected by Gödel's work" (Dawson 1988: 90).

Gödel's first proof relies on arithmetized, self-referential metamathematical statements about provability and on Cantor's controversial, platonic, diagonalization argument. The self-referential statements permit interpretational ambiguities and paradoxes to creep in. Cantor's diagonal argument, presented lucidly in Beckman (1980), counter-intuitively asserts that some infinite sets are larger than others. The second proof, which denies the provable consistency of arithmetic on natural numbers, relies on the nonconstructive logic of the first theorem. The logic of these nonconstructive existence proofs is convoluted and difficult to follow (see Nagel, Newman & Gödel 1958 or Beckham 1980 for reasonable expositions). More straightforward is Turing's formally equivalent Halting Problem argument, discussed in the next section.

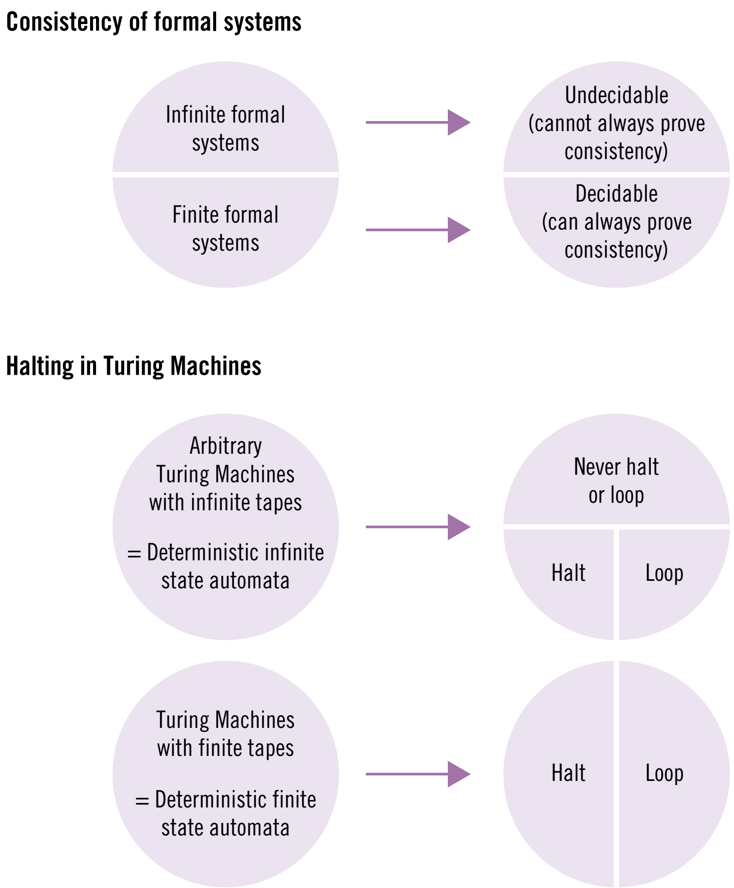

Figure 1: Finitist interpretations of Gödel's Proof and Turing's Halting Problem.

Gödel's proof could be rejected on constructivist and/or finitist grounds. In his *Remarks on the Foundations of Mathematics* Wittgenstein disputed the platonic methodological assumptions and interpretations embedded in the theorems that essentially altered the meaning of "proof" (Shanker 1987, 1988; Wittgenstein 1978; Kielkopf 1970; Wright 1980, Rodych 2011). Because Wittgenstein often expressed his ideas aphoristically and did not make explicit his interpretive framework, there has been an ongoing, complicated exegesis-laden philosophical debate over what he meant. It is possible that more publications have been devoted to whether (or in what sense) Wittgenstein was a strict finitist than to the theory of strict finitism itself or the validity of its critique.

One could argue, from a strict finitist perspective, that the introduction of actual and potential infinities into a formal system inherently produces inconsistencies that can be avoided by restricting oneself entirely to finite sets (ultra-finitism). Unlike potentially-infinite systems, the consistency of finite systems can be proven because all the possible constructions can be enumerated, evaluated, and compared. We can have provably consistent formal systems if we keep their scope finite. This interpretation of the meaning of Gödel's Proof and Turing's Halting problem is summarized in Figure 1. This is why the consistency of the computations we actually carry out is unproblematic from a foundational viewpoint

– physically realizable, finite formal systems are qualitatively different from their potentially-infinite and actually-infinite counterparts. The latter interpretation and the prescription to stick to finite systems is almost never discussed in the literature, mainly because most mathematicians and philosophers of mathematics have strong Platonist intuitions that they loathe to question, let alone give up entirely.

Decidability and the halting problem

A concrete example of how finiteness changes the nature of conclusions about consistency and decidability can be shown by examining the effect that finiteness has on the outcome of the halting problem. The halting problem is a decision problem in which the question posed is whether a single algorithm (computational process) always leads to a correct yes or no answer. Turing posed the question of whether, for any arbitrary Turing machine, given a particular set of inputs, it is possible to determine whether the machine will eventually halt.

Turing machines consist of a machine head that is a deterministic finite state automata (FSA), a tape of discrete locations that contains their external inputs for each computational step, either a 0 (blank) or a 1 (mark), and mechanisms for reading and writing symbols on the tape, and for moving the tape one location in either direction (Beckman 1980). Depending upon the symbol on the tape and the internal state of the FSA, the Turing machine decides its next action (read, write 0, write 1, move tape left, move tape right, do nothing). The tape is indefinitely extendable, so that the Turing machine can construct and handle arbitrarily large numbers and other potentially-infinite sets of elements. The computational capabilities of the Turing machine, with its finitary, deterministic finite-state automaton computational rules and its potentially-infinite tape parallel finitary procedures on natural numbers, are such that the halting decision problem is formally equivalent to Gödel's incompleteness proofs (Kleene 1967; Davis 1958).

A Turing machine is therefore specified by the state transition rules of its machine head and its initial conditions or input string, i.e., the pattern of 0's and 1's on its tape. Its current total machine state is specified by the state of the tape, i.e., pattern of 0's and 1's on the tape, the state of the machine head, and the position of the read/write head on the tape. The Turing machine has three possible behaviors:

1 | reach a terminal total machine state and halt,
2 | fall into a computational loop in which it endlessly repeats a finite sequence of total machine states, i.e., it never halts, or
3 | transit over an endlessly expanding, infinite set of locations on the tape, and never repeat a total machine state, i.e., it never halts.

Being a machine whose behavior is completely determined by its total machine state, i.e., a "trivial machine" in Heinz von Foerster's terms, if a total machine state is ever encountered a second time, it will loop.

Along lines parallel to Gödel, Turing argued that, for any arbitrary Turing machine, there is no general, finite algorithm that can determine in a finite number of steps whether the machine will eventually halt (Turing 1936). But if the Turing machine has a tape of finite length, then there are a finite number of total machine states available to it. In this case, one can always determine within a finite number of steps whether the machine will halt or not. The finite tape eliminates the possibility of an ever-widening set of tape locations, so the halting problem becomes one of deciding whether the machine will enter a terminal state (halt) or a loop (never halt). One simply needs another machine that keeps track of the total machine states of the first and records whether the machine has either halted or repeated a previous total machine state. Even simpler, if one counts the number of states traversed, and this is larger than the number of total machine states of the first machine, then the machine must have repeated a total machine state, and therefore will never halt. Thus, for finite state automata, as opposed to arbitrary Turing machines with potentially-infinite total machine states, the halting problem *is* decidable.

These profound differences between finite and potentially-infinite machines tend to be completely overlooked both in the foundations of mathematics and in theories of formal computation. Most often the behaviors of finite state automata and finite formal systems are regarded as trivial and uninteresting, but sometimes the neglect is due to a conflation of finite and infinite automata. This arises because every finite state automaton is a Turing machine, but not every Turing machine is a finite state automaton. It is those infinite state Turing machines that do not loop and do not halt that do not have finite state automaton equivalents. It is thus easy to conflate in one's mind finite state automata (computers) and the set of "arbitrary Turing machines," despite the nonequivalence of the two sets.

These kinds of conceptual errors arise when one tries to discuss the limitations of physically-realizable ("real world") computers with dyed-in-the-wool platonists. Physically-realized computers, of course, have finite numbers of states and are therefore finite state machines (Cariani 1989). In Cariani (1992), I argued along these lines, i.e., that computer simulations necessarily have closed state spaces of possibility and consequently received hostile responses from outraged reviewers who argued that:

1 | computers are Turing machines (correct),
2 | arbitrary Turing machines are open-ended because their behavior is not decidable (correct), and therefore
3 | computers and computer simulations are unpredictable and open-ended (incorrect).

These kinds of conflations between finite and infinite systems, along with invocations of Gödel and Turing, have long been a part of the intellectual landscape of mathematical physics, computer science, artificial intelligence, and even artificial life (in addition to errors of more technical sorts, cf. Franzén 2005). They are easily found on the current pop-science scene, where whole cottage industries have been spawned to harness the interpretation of Gödel (Penrose 1989). This species of quasi-religious rhetoric is likely to remain with us far into the future.

Because real-world computers are finite state devices, computability (limitations of decidability in infinite systems) has absolutely no relevance for real world computing. On the other hand, computational complexity *is* highly relevant for practical, real-world computing because the numbers of available states (memory) and computational steps (processing speed) determine what algorithms can be computed in practice. Limitation of our concept of computation to finite state machines thus corresponds much better to how we use computations and computational devices in real life.

To return to the *New Yorker* cartoon, as regards consistency, the finite arithmetic systems that we use in everyday life are fundamentally different from the infinite systems that are contemplated in pure mathematics and philosophy. It is only if one believes that the latter has something to do with the former that one can be troubled by a crisis in the philosophical foundations.

Finitism

We have seen that radical constructivism is naturally allied with varieties of mathematical psychologism, constructivism, and formalism. In the radical constructivist psychologist mode, one can conceive of mathematical concepts as mental structures constructed through experience, action, and thought. In its mathematical constructivist mode, mathematical objects are actively constructed by sequences of concrete overt actions that parallel the means by which objects and object systems are constructed mentally (e.g., numbers and counting operations). In the formalist mode, the manipulation of concrete tokens according to socially-shared conventions to reach reliable intersubjectively-verifiable agreement parallels radical constructivist conceptions of cooperation, communication, and "shared realities."

We have also considered where radical constructivism, in its epistemologically-grounded rejection of infinities, might part company with Hilbert's formalist program, whose aim was to tame infinities. For several reasons, the denial of imperceptible infinities effectively commits radical constructivist philosophy of mathematics to some form of finitism. Finitism denies the reality of infinite sets. Some forms of finitism deny only actual infinities (e.g., the uncountably infinite set of irrational numbers) but admit countably-infinite sets, such as the natural numbers (classical finitists including Markov and Kronecker, who famously said that "God made the integers; all else is the work of man"). Some deny countably infinite sets and potentially-infinite constructions because these can never be completed (strict finitists). In these terms, Aristotle would be considered to be a strict finitist.

Still other, more conservative varieties deny those numbers for which we do not yet have the means at hand to deal with them as individual entities. For very large numbers whose representations far exceeds the capacity of our present and future representational systems (e.g. numbers of digits larger than the estimated number of fundamental particles in the universe), we lack the means of actually constructing, reliably manipulating, and/or determining some of their numerical properties (e.g., how many 7's are in their base ten representation). These varieties of finitism come under the rubrics of feasible numbers, actualism, Esenin-Volpin's ultra-finitism, and "fanatical finitism" (Mawby 2005). Under the notion of attainable numbers, we have as many numbers available to us at any time as we have the mental or mechanical means to distinguish and manipulate as individuals. Thus, as our computers grow in capability, the envelope of attainable numbers progressively expands (i.e., a "Chuck Yeager" theory of numbers that keeps on breaking the current "number barrier" as machines successively improve).

Actualism has the advantage of describing concretely the domain of numbers that we utilize in real life. Although we may have generative, computational descriptions for astronomically large numbers, we do not know them as individual integers. A striking example, presented in Beckman (1980), is $f(4,4)$ of the doubly recursive Ackermann's function, which consists of three lines:

1 | $f(0,n) = n + 1$
2 | $f(m,0) = f(m-1,1)$
3 | $f(m,n) = f(m-1, f(m,n-1))$.

Beckman notes,

> "It thus appears that this function is effectively computable, and we can compute its value for any given pair of arguments. However, [Ackerman's function] is a most remarkable function, and perhaps we should examine more closely our use of the phrase 'we can compute.' To find, say $f(2,2)$ is a trivial computation and $f(3,3)$ (=61), can be computed in a fraction of a second on any modern computer. But $f(4,4)$ is another story. … If every one of these [10^{80}] particles [in the known cosmos] were used in some way to represent one digit in the decimal representation of $f(4,4)$, not only would they not be sufficient, but *we would not even be able to represent that number which is the number of digits in $f(4,4)$* (i.e.,[$\log_{10} f(4,4)$]). What meaning then is there in the statement that 'we can compute' this function?" (Beckman 1980: 128)

So, in what senses can we reasonably say that this number exists? It certainly exists in our minds, as a set of concepts. It also exists as the end product of a concrete recursive computational procedure, but it does not yet exist as a unique integer whose relational numeric properties can be interrogated (e.g., is it divisible by 5?). If we stick to numbers that can be reliably manipulated and sets that are constructed from concrete collections of individuals (Goodman 1956), then we can be reasonably assured of the reliability of our computations and our logics.

We tend to believe that these philosophical differences are intellectual, but perhaps they simply reflect underlying differences in temperament. In religious terms, platonists are the mystics, dreamers, true believers, and deists, with varying degrees of certainty and piety. Formalists like Hilbert are agnostics, who nevertheless would like to believe in a benevolent god if one could be proven, even indirectly, by rational reasoning. Constructivists withhold belief until they see concrete evidence. Finitists are the skeptical agnostics and atheists of the group, denying the practical relevance of infinities and/or their existence (apart from in the minds of platonists). In these terms, radical constructivists are naturally skeptical atheists when it comes to platonic mathematics, realist physics, and all philosophical theories that disconnect reference, truth, and meaning from real observer-actors (e.g., Putnam's famous assertion that "meaning is not in the head").

Finally, one can harbor strong doubts that the metaphysics of mathematics has any practical relevance. The foundations debate is so abstract and abstruse that it conjures medieval scholastic debates over properties of angels. In the end it makes little difference for working mathematicians whether they believe in the absolute reality of their mathematics – in any case their theorems must be unambiguously demonstrated to others in the form of a proof. On the other hand, we believe that there is inherent utility, for both mathematicians and the rest of us, in a clearer conception of the nature of mathematical constructs. Further, we believe that a psychologically- and neurally-grounded radical constructivist theory of how mathematical concepts are constructed by autonomous human subjects, along the lines of Lakoff & Núñez (2000), Dehaene (2011), and Glasersfeld (1991, 2006), could play a very positive role in developing educational strategies that facilitate mathematical creativity and thought.

Conclusions

In dealing with wide ranges of philosophical and scientific questions, it is useful to be able to identify patterns of thought and see quickly where their metaphysical premises lead. For example, Chomskian notions that languages are infinite constructs, that ideal grammatical form is the essence of language, and that the communicative use of language is of secondary concern, can be immediately recognized as a platonist research program. We see platonic thought pervading the possible-worlds "multiverse" interpretation of quantum mechanics (a metaphysical hypothesis that is inherently empirically untestable) and in possible-worlds semantics. Platonist instincts stoke belief that time travel may be possible because some mathematics of physics allow it.

The answer is not to ban such platonic notions from our discourses. As wacky as they sometimes seem, they may eventually lead to new ideas that are practically testable. Instead we need to learn to demarcate clearly those ideas that can have only heuristic value from those that can (eventually) be soberly evaluated. The challenge is to learn to use each kind of idea in an appropriate way, as an aid to the imagination or as a testable hypothesis. In this way, radical constructivism can provide a guide for thought that is both creatively open to latent possibilities and sensibly critical of those ideas that do not lend themselves to practical evaluation.

Radical constructivist thinkers in the future will eventually develop theories of the nature and creation of mathematical objects in minds and brains, and in the process perhaps demystify the nature of mathematical ideas. We think that it is a only matter of time until the present platonic wave runs its course, and that the world comes around to see the need for a new way of looking at mind, matter, and mathematics (Cariani 2010).

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