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Do young children grasp the inverse relationship between addition and subtraction? Evidence against early arithmetic

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Abstract

This research investigates young children's reasoning about the inverse relationship between addition and subtraction. We argue that this investigation is necessary before asserting that preschoolers have a full understanding of addition and subtraction and use arithmetic principles. From the current models of quantification in infancy, we also propose that the children's earliest ability to add and subtract is based on representations combining and separating sets of objects without arithmetical operations. In an initial study, 2- to 5-year-old children was tested on addition ($2 + 1$), subtraction ($3 - 1$) and inversion problems ($2 + 1 - 1$) by using Wynn's procedure (1992b) of possible and impossible events. Only the oldest age group (4–5 years) succeeded on the inverse problem. In a follow-up study, 3- to 4-year-old children were given a brief training intervention in which they performed adding and subtracting transformations by manipulating small sets of objects without counting. The beneficial effects of the training support the claim that preschoolers respond to the inverse problem on the basis of object representations and not on the basis of numerical representations.

Keywords: Simple arithmetic; Addition; Subtraction; Inverse relation

Over the past two decades, many investigators have studied the numerical reasoning capacities of preschool children and considerable progress has been made in this domain (e.g., Cooper, 1984; Gelman & Gallistel, 1978; Ginsburg, 1983;

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Hughes, 1986; Huntley-Fenner & Cannon, 2000; Huttenlocher, Jordan, & Levine, 1994; Levine, Jordan, & Huttenlocher, 1992; Siegler & Robinson, 1982; Sophian & Adam, 1987; Starkey, 1992; Wynn, 1992a). It has been found that young children can carry out numerical reasoning involving simple addition and subtraction by 2 or 3 years of age, provided that the numbers involved are small. They can solve calculation problems such as "$2 + 1$" and "$3 - 1$" in which addition or subtraction is carried out on real objects which children can see, but where the final total is either hidden in some way or displayed by using a violation-of-expectation paradigm.

Thus it would seem that the ability to reason about numerical transformations appears early in life, well before children receive explicit education in mathematics in school. But this prompts the question of the nature of reasoning involved in numerical transformation tasks. In this paper, we argue that young children's knowledge about numerical transformations is not arithmetically based but object-based. The hypothesis that the performance of preschoolers does not entail arithmetical reasoning is based on two assumptions. The first is that young children do not manipulate addition and subtraction in numerically meaningful ways. The second is that young children use a non-numerical mechanism to solve addition/subtraction tasks similar to those of infants. These two assumptions are discussed before presenting empirical data to support our hypotheses.

1. What numerical knowledge underlies arithmetical reasoning?

Even if the ability to appreciate the effects of numerical transformations emerges early in childhood, can we say that young children have an exact understanding of addition and subtraction in the fullest sense of the word *arithmetic*? Piaget (1952) argued that someone who can add and subtract correctly does not necessarily understand addition and subtraction because the additive and subtractive operations must be coordinated to be really assimilated. Thus, as Bryant (1992) asserted, before asserting that young children use addition and subtraction in an arithmetical sense, it may be necessary to check that additive and subtractive transformations are well-known as inverse transformations of each other. For Piaget and Moreau (1977), children do not become aware of the inverse relationship between addition and subtraction before the level of *concrete operations*.

Even if reversibility of displacement is acquired by 18 months (Piaget, 1952b), reversibility of numerical transformations is not grasped before 6 or 7 years old. Furthermore, Piaget (1967) introduced a major distinction between "true reversibility" based on the logical operativity and "empirical reversibility" sometimes called *renversibilité* or "empirical return". Two different motor activities (or perceived events) may be juxtaposed and seem to be the reverse of each other, but this is remote from the logical reversibility in which they both function (e.g., addition and subtraction in a group structure). In other words, the empirical return — contrary to reversibility — is not the same operation as the direct operation but in the opposite direction: it is a successive action (or transformation) that is qualitatively different from the former.

When children understand both addition and subtraction as interdependent operations rather than as independent transformations, they can reason arithmetically by using the relations between parts and wholes and the additive composition of numbers. As long as the representation of addition is not coordinated with the representation of subtraction, the performance of young children in calculation problems cannot result from an arithmetical reasoning as such. At most, these children are able to represent the result of numerical transformations regardless of addition and subtraction operations. The hypothesis of arithmetical reasoning is untenable when a deficit in conceptual understanding such as addition and subtraction are represented as two separate operations that exclude each other.

The idea that very young children might understand the inverse relationship between addition and subtraction has rarely been challenged empirically. So far the studies conducted to test or to induce the inverse property have been carried out only on children from the age of 5 or 6 (Beilin, 1965; Denney, Zeytinoglu, & Selzer, 1977; Field, 1981; Lifschitz & Langford, 1977; Wallach & Sprott, 1964; Winer, 1968; Wohlwill, 1959; and more recently, Bisanz, LeFevre, & Gilliland, 1989; Bryant, Christie, & Rendu, 1999; Stern, 1992). Only one study (Starkey & Gelman, 1982) reported an attempt to test 3-year-old children's understanding of inversion. Children were asked to solve inverse problems in the form of $a + b - b$. But, as Bryant et al. (1999) pointed out, the results invalidate the authors' conclusion that young children understand that addition and subtraction cancel each other out because the children solved the inverse problems (e.g., $3 + 1 - 1$) as well as the control problems (e.g., $2+2-1$). It is therefore highly probable that they computed the result of additions and subtractions without the help of the inversion principle. If this had not been the case, they would have succeeded in solving the inverse problems better than the control problems. More recently, Bryant et al. (1999) demonstrated that the scores of 5- and 6-year-old children were higher on inverse problems than on control problems. Their results prove that after age 5 years, at least, children are able to use the inversion principle in a genuinely quantitative way. Does this mean that before age 5 years children lack this principle? Maybe so, but two objections are possible.

The first is that in Starkey and Gelman's study, young children were asked to produce a verbal response to the inverse and control problems. So, even if the range of problems were presented with concrete material, the use of the inversion principle could be inhibited by this verbal requirement. It is well known that verbal reports are problematic in young children. Their ability to reason about numerical transformations have been forcefully demonstrated with nonverbal tasks.

Recently, an experiment was designed to dispel this objection (Vilette, 2002). Eighty-nine 2- to 4-year-old children were tested both on the inverse and the control problems by using a nonverbal task in which they had to construct an array that is numerically equivalent to another, hidden array. This task requires them to determine how many objects are in the hidden array after objects have been added to and/or taken away from it. The results clearly indicated no difference in the number of correct answers in the inversion and control problems: the control scores ranged from .03 to .64 according to age groups, with a median of .29, and the inverse scores ranged from .03 to .56 with a median of .31. Consequently, the results confirm that young children fail to use the inversion principle and demonstrate that this failure is not due to the verbal measure of inversion.

The second objection is more subtle. In inversion studies, children are faced with control problems which necessarily require them to compute the exact outcome of numerical transformations. But this requirement is confusing precisely because children must not compute in the inverse problems. The trouble is that young children are inclined to compute in all problems even if they understand the inversion principle. They may choose to compute systematically because, for example, it is difficult to change strategy in the course of the evaluation, or because they need to verify the result of the numerical transformations. Hence the comparable performance in control and inverse problems. We are then faced with the difficulty of evaluating whether or not young children actually realize — without numerical computation — that addition and subtraction cancel each other out. An initial experiment addressed this difficulty. Children were asked to make a judgment (normal or not normal) about the effects of perceived transformations (addition, subtraction and inversion) with a possible and an impossible outcome and not to compute the numerical values of such transformations. Before developing this approach, we will discuss the mechanisms that may underlie children's reasoning in nonverbal calculation tasks.

2. What mechanism underlies preverbal quantification?

Although our present concern is not directly with infants' numerical competencies, we must refer to the current models of early quantification in order to specify the nonverbal mechanism of quantification that operates in childhood. Young children's quantification necessarily has its roots in infancy. Furthermore, if the calculation mechanism is already present in infancy, it should be evident and easily available in young children (Huttenlocher et al., 1994). Two models of early quantification have been proposed in the literature: the accumulator model and the object-file model. These models, which are radically opposed about the nature of early numerical representations, will be sketched and compared (for a more detailed account of these well-known models, see Koechlin, Dehaene, & Mehler, 1997; Mix, Huttenlocher, & Levine, 2002; Uller, Carey, Huntley-Fenner, & Klatt, 1999).

The accumulator mechanism presented by Gelman (Gallistel & Gelman, 1992, 2000) provides mental representations of numbers in the form of analogical magnitudes (e.g., "_" being a representation of 1, "__" a representation of 2; "___" a representation of 3, and so on). The numerical values are given by the ordinal position of the accumulator ("_, __, ___, . . . "). In fact, this mechanism is a specific realization of Gelman's original model (Gelman & Gallistel, 1978) in which numerosities are represented by a list of arbitrary symbols called "numerons" (e.g., "!, #, &, $, . . . " or "?, ¤, §, @, . . . "). In both systems, a counting process determines the numerical representation in such a way that 2, for example, is represented by the magnitude "___" or the numerons "#" or "¤". Given that the mechanism can create several accumulators with fullness values stored in memory, different sets of entities can be counted and compared.

Arithmetical reasoning would then be based on the idea that operations carried out on the magnitudes or the numerons are isomorphic to operations of conventional arithmetic (Gallistel & Gelman, 1992, 2000; Wynn, 1995, 1998). Addition, for example, could be achieved by "pouring" the contents from an accumulator representing one value into an accumulator representing another value, or more specifically, by transferring the contents of two accumulators into a third empty accumulator. Subtraction could be similarly achieved by removing the content of one accumulator until the right amount corresponding to another accumulator has been removed, or by creating new accumulators to avoid loss of the initial value in the subtraction operation. In both operations (addition or subtraction), a matching process of any of two symbols produced by accumulator mechanism would indicate whether the represented numerosities are the same or different. The isomorphism postulated between accumulator operations and arithmetic operations constitutes the critical condition of early arithmetical reasoning.

The object-file model is an alternative view which does not require such arithmetic knowledge in infants. According to this model, each perceived object is encoded in a separate file with its coordinates in space and time and its specific properties such as color, size, direction, location. In this view, a new file object is opened every time a physical object appears at a novel location. As Koechlin et al. (1997) pointed out, the critical difference between the accumulator model and the object-file model concerns the nature of the representation of numerosity: it is a representation of numbers in one case, and a representation of objects in the other. Thus, in the object-file model, an array of two physical objects would be encoded by a representation of the form "O_i O_j" or "X_i X_j", or any other representation given that there is no single symbol for 2 at all, not "___" or "#" or "¤" nor any other. Consequently, according to the object-file model, there is no calculation procedure in the infant's mind but only an ability to manipulate mental representation of objects. As modeled by Simon (1998), reasoning about numerical transformations would then be based both on the process of one-to-one correspondence and on the immediate or very short-term memory of object permanence (see also Bideaud, 1995, for a similar account).

Although there is no conclusive proof for deciding between the two models of infant number representation, several arguments favor the object-file model (Uller et al., 1999). Two main arguments can be taken up here to justify the idea that precise quantification in young children is also done without any knowledge of arithmetic. Firstly, there are the massive effects of set size on children's performance. In all studies (e.g., Cooper, 1984; Ginsburg, 1983; Hughes, 1986; Levine et al., 1992; Siegler & Robinson, 1982), researchers agreed that 2- or 3-year-old children only solve calculation problems on the first three numbers, or even on numbers 1 and 2 (Starkey, 1992), with numerical transformations of one object. It is not before ages 3 or 4 years that children grasp precisely transformations of two objects or more. But even at age 4 years, all the problems involving small numbers (<5) are still not solved by most of the children.

Children's difficulties with calculation problems always increase significantly with the numerosity of the result and the numerosity of transformation. These dramatic effects of numerosity are best predicted by the use of one-to-one correspondence (object-file model) rather than by the use of a matching process (accumulator model). Indeed, the process of one-to-one correspondence detection becomes much more complex as the number of objects in the two representations increases. This is because the memory load also increases with the number of objects to be compared. This is not the case for the matching process because the comparison between two numerical symbols, whatever the number of objects they represent, is made by noticing a numerical mismatch/match between them. So, remembering one particular symbol does not require more memory resources than remembering another symbol, and discrepancy detection varies slightly with the content of the symbols.

Even if discrimination of analogical representations becomes harder with the size of the number, according to Weber's law (for example, "____" vs. "______" is harder to discriminate than "_" vs. "__"), the accumulator model is hardly compatible with the numerosity effects observed on small sets of 1–3 objects. These effects are easier to account for with the object-file model owing to the fact that difficulty and error probability in applying the one-to-one principle start from the first correspondence and gradually increase with the following correspondences.

Another argument concerning the role of spatio-temporal clues in object perception also provides support for the object file model. Uller et al. have demonstrated that infants' reactions to an impossible outcome following addition and subtraction operations (Wynn, 1992b) are not observed when clues about object location are unspecified or sharply reduced. In other words, infants must concretely see where the objects are placed by the experimenter before grasping the effects of simple adding and subtracting. What is interesting in this respect is that young children also fail addition and subtraction tasks when these take place in contexts where there is no reference to specific objects (Hughes, 1986). It is not before ages 4 or 5 years that children solve calculation problems in a context of hypothetical objects which are not perceived. Although this fact does not provide conclusive evidence for one model or the other, the key role of object perception lends empirical support to the object-file model and not to the accumulator model because the main properties of objects, such as their location and their physical appearance, are encoded first in the former but not in the latter.

The two previous arguments lead us to think that the calculation mechanism in young children as in infants might also be based on the process of one-to-one correspondence and the permanence of three or four objects in working memory.

This hypothesis, which rejects the arithmetical nature of children's reasoning in calculation problems will be tested in a second study. Recall that the first study assessed the ability of young children to make judgments about the results of inverse transformations on small sets without computing the numerical values of such transformations. If children really have a full understanding of addition and subtraction, they should be able to use the inverse relationship between addition and subtraction. If this is not the case, we cannot grant them knowledge of arithmetic. The second study asks whether children's reasoning about inverse transformation may improve after a brief training intervention. Different training procedures have been designed to decide which of the two models of young children's quantification might be correct (accumulator model or file-object model).

3. Spontaneous quantification

Children's ability to reason about numerical transformations were examined by means of a method adapted for preschoolers within the context of possible or impossible events paradigm. We used a Wynn-type procedure similar to Wynn's (1992b) except that the children were asked to verbally indicate whether the results they saw were "normal" or "not normal". This procedure has already been fruitful in other experiments with young children (Houdé, 1997; Vilette & Mazouz, 1998). Several reasons motivated the choice of this procedure.

As we have seen, the difficulty is to lead children to reason about inverse transformations without computing the numerical values of such transformations. If the Wynn-type procedure cannot guarantee that the children do not compute, it does not prompt them to do it. The reasons are twofold: an exact quantification is never required as it is, for example, in a production task (Starkey, 1992; Vilette, 2002); and no numerical value is mentioned during the procedure. Children are only asked whether what they saw is normal or not. In addition, if children really calculated the result of numerical transformations, one would not find differences of performance between addition, subtraction and inversion since all problems involve the same small numerosities (cf. as follows).

Of course, the above arguments do not prove decisively that children do not compute the result of inverse transformations. But if children understand the inverse relationship between addition and subtraction, the Wynn-type procedure would have to lead them to use this knowledge. In that case, their performance on the inverse problem should not be completely hopeless compared to their performance on the addition and subtraction problems.

Admittedly, we may invoke the highest load in working memory for resolving inverse problems because children must deal with two successive operations and not only one as in addition or subtraction problems. But neither the accumulator model nor the object-file model raise this limitation. Only the numerosity of result or the numerosity of transformation can increase the memory load and the difficulty of the task. In both models, the quantification processes are not limited to representing only one operation in time and space. There is no reason why a quantitative representation can not be modified by two successive transformations.

For all the above reasons, we suppose that the violation-of-expectations procedure permits testing young children's use of the inverse relation without numerical calculation. So three age groups were given three problems: (1) the addition problem "$2 + 1 = 3$" with impossible event "$2 + 1 = 2$", (2) the subtraction problem "$3 - 1 = 2$" with the impossible event "$3 - 1 = 3$", and (3) the inverse problem "$2+1-1 = 2$" with the impossible event "$2+1-1 = 3$". The numbers involved in each problem were chosen for three reasons. First, they are smaller than 4 and, therefore, in the numerosities range that very young children can grasp by direct perception (subitizing) or by a counting scheme (Starkey & Cooper, 1980). Second, the addition and subtraction problems involving other small numbers, such as $1 + 1$ or $2 - 1$, are used in a preliminary phase to familiarize the children with the material and the procedure. Third, in order to have equivalent problems compared to numerosity, the final total in all problems is either 2 or 3.

Three age groups ($n = 22$ each), 2-year-olds ($M = 2.5$; range 2.2–2.8), 3-year-olds ($M = 3.5$; range 3.2–3.8) and 4-year-olds ($M = 4.6$; range 4.2–4.9), were tested. The younger children were recruited through 4-day care centers and the older ones through three preschools they attended. All came from a middle-class population and from homes where French was the primary language.

The children were shown a wooden puppet theater, 66 cm high, 70 cm wide, and 30 cm deep, resting on top of a table. A revolving screen allowed us to either reveal or occlude the stage. A side-window allowed the experimenter to add or take away objects in such a way that the children could see what was happening (addition, subtraction) without seeing the configuration hidden by the screen. A trap door behind the stage allowed the objects to be manipulated surreptitiously when the screen was raised. Instead of Mickey Mouse dolls (Wynn, 1992a, 1992b), the objects presented to children were the well-known French Babar dolls (20 cm high and 10 cm wide).

Children were tested individually in a quiet location in a single testing session that lasted about 15–20 min. The experimenter began by introducing the Babar dolls and testing the instructions in two "familiarization" events, one for "$1 + 1$" addition, another for "$2 - 1$" subtraction. For "$1 + 1$" addition, the children were asked to look at one Babar on the stage. Then, the experimenter raised the screen and conspicuously put another one behind it. Next, the screen was lowered and the children were shown two or one Babar dolls. They were asked whether "that's normal" or "that's not normal" (the French sentences requested were "ce n'est normal" or "ce n'est pas normal"). After each trial, a verbal feedback was given ("that's right" or "that's not right"). The same procedure was repeated for "$2-1 = 1$" subtraction with the impossible event "$2 - 1 = 2$".

After the familiarization phase, the experimental phase was started and consisted of three problems ("$2+1$"; "$3-1$" and "$2-1+1$") with one possible event and one impossible event. The order of the problems was invariant across children and over age. The order of possible and impossible events was intermixed to avoid

Table 1
Experimental trials and order of presentation

Initial display	Transformation realized	Possible event	Impossible event	Order of presentation
● ●	+ ●	● ● ●	● ●	P–I
● ● ●	− ●	● ●	● ● ●	I–P
● ●	+ ● − ●	● ●	● ● ●	P–I

Table 2
Proportions of correct responses on all three problems in each age group

	Addition, $2 + 1$	Subtraction, $3 - 1$	Inverse, $2 + 1 - 1$
2.5 years ($N = 22$)	.64	.14	.05
3.5 years ($N = 22$)	.73*	.45	.23
4.5 years ($N = 22$)	100**	.91**	.91**

* $P < .026$, one-tailed.
** $P < .001$, one-tailed.

response bias. Table 1 shows the six problems successively administrated to each child.

The procedure was identical for the three problems. For example, in the first trial (i.e., the addition problem "$2 + 1 = 3$") the initial display (two Babar dolls) was presented for 5 s and the children were asked to look carefully at the stage and at what was happening. After having looked at the operation (the addition of one Babar doll), they were asked whether what they saw (three Babar dolls) was "normal" or "not normal"[1]. No verbal feedback was given.

Children's responses to each problem was scored as correct or incorrect on the basis of their answers on both possible and impossible events. They were credited with a correct response (1 point) if and only if their answers were correct on both the possible ("ce n'est normal") and impossible ("ce n'est pas normal") events. In all others cases, the responses were scored incorrect (0 point). These criteria avoid the response bias of subjects always giving the same answer irrespective of the outcome of the event.

Do young children use the inverse relationship between addition and subtraction? In the two youngest age groups (2.5- and 3.5-year-olds), most children failed the inverse problem (21 and 22 subjects on a total of 22 in each group, respectively); whereas in the oldest age group (4.5-year-olds), most children succeeded on the inverse problem (20 on a total of 22). Table 2 shows children's proportions of correct responses to individual problems in the three age groups. For each condition, a *binomial test* was conducted to determine if children succeeded on the

[1] Note that no additional justification was requested because in previous experiment (Vilette & Mazouz, 1998), we have observed that this request inhibited children's behavior.

addition, subtraction and inverse problems more often than expected by chance. The 2.5-year-olds' performance was at chance level on all three problems. The 3.5-year-olds were successful on the addition problem, but their performance was still at chance on the subtraction and the inverse problems. Only the 4.5-year-olds responded correctly above chance on all three problems.

We next turned to the question about the relative difficulty of addition, subtraction and inverse problems. A non-parametric analysis was conducted to compare the performance on all three problems. In our total sample ($N = 66$), 52 children responded correctly on the addition problem, 33 children on the subtraction problem and 26 on the inverse problem. Performance was thus significantly better on the addition problem than on both the subtraction ($X^2(1, N = 66) = 6.75$, $P < .0094$) and the inverse problems ($X^2(1, N = 66) = 11.08$, $P < .0009$). No difference was found between subtraction and inversion. Similar results was obtained for the 2.5-year-olds (addition vs. subtraction: $X^2(1, N = 22) = 6.75$, $P < .0094$ and addition vs. inverse: $X^2(1, N = 22) = 11.08$, $P < .0009$) and the 3.5-year-olds (addition vs. subtraction: $X^2(1, N = 22) = 4.17$, $P < .0412$ and addition vs. inverse: $X^2(1, N = 22) = 9.09$, $P < .0026$). On the other hand, performance by the 4.5-year-olds was not different on the three problems.

An error analysis was also conducted to determine whether errors on the inverse problem was due to a response bias. For example, children may have opted for the outcome that was different from the initial display. This would lead to errors on the inversion problem but success on the addition and subtraction problems. In this specific case, the predominant type of error pattern would be "111100" (where 1 and 0 designate correct and incorrect reactions respectively on the possible and impossible events for the addition, subtraction and inverse problems). The analysis revealed no predominant type of error pattern both for the 2.5 year group and for the 3.5 year group. Error patterns were not systematic. Moreover, even if we consider responses on each problem alone (i.e., addition, subtraction or inverse) no error pattern ("00", "10" or "01") was predominant in the two age groups.

Finally, three results should be underlined. All children who succeeded on the inverse problem ($n = 26$) also succeeded on both the addition and subtraction problems. Seven children responded correctly on both the addition and subtraction problems and incorrectly on the inverse problem. Except for one child, all children who responded correctly on the subtraction problem also responded correctly on the addition problem.

The major finding was the failure of 2.5- and 3.5-year-old children on the inverse problem. Only 6 out of 44 children performed correctly on the inverse problem. This finding is very critical in regard to a full understanding of addition and subtraction in preschool children. It does not fit well with Gelman's accumulator model and the claim of early arithmetic.

Even the 4.5-year-olds performance was not conclusive in this regard because they succeeded all three problems. Only 2 out of 22 children failed the subtraction and the inverse problems. Children of this age may represent the precise result of numerical transformations. However this capacity may be based on the process of one-to-one correspondence and the objects permanence as proposed by the object-file model which need not grant arithmetic knowledge to young children.

Poor performance on the subtraction problem by 2.5- and 3.5-year-olds may appear flabbergastingly astonishing: only three 2.5-year-olds and ten 3.5-year-olds performed successfully on the subtraction problem. Furthermore, if children succeeded better on the addition than the subtraction problem, the lack of near-perfect performance on the addition problem may also appear inconsistent with previous studies (e.g., Cooper, 1984; Starkey, 1992; Vilette, 2002) suggesting that very young children can solve calculation problems with small numbers. However, poor performance on addition and subtraction corresponds precisely to the performance expected if children do not compute the outcome in our judgment task as they do in a production task. Furthermore, even if the greater difficulty of subtraction compared to addition is rarely emphasized (see however, Kamii, 1990; Uller et al., 1999), it is well known that that additive transformations are easier to grasp for young children than subtractive transformations (Cooper, 1984; Fuson, 1988; Siegler & Robinson, 1982; Vilette, 2002; Vilette & Mazouz, 1998; see however Huttenlocher et al., 1994; Starkey, 1992). As pointed out by Kamii (1990), this may be because addition is more natural than subtraction, and more generally because cognition and action first operate positively and deal with the positive aspects of situation before they deal with the negative aspects.

Overall, the results corroborate Piaget's (1952a, 1952b) contention that children can add or subtract correctly without understanding the inverse relationship between addition and subtraction. If so, then young children do not reason arithmetically since the inverse property defining the system of arithmetic is not represented. Contrary to what has been claimed by the accumulator model, it is likely that preschoolers' reasoning on the numerical transformations are object-based and not number-based. A second study tested this hypothesis.

4. Provoked quantification

In order to verify that preschoolers' reasoning on numerical transformations are object-based, young children were tested on the inverse problem before and after a brief exercise in which they were given the opportunity to make adding and subtracting transformations without mentioning numerical value. Only children who succeeded on both the addition and the subtraction problems but failed the inverse problem (baseline criterion) were selected to participate in the study. We expected that correct responses on the inverse problem may increase following a training session enabling them to perform adding and subtracting transformations in concrete situations.

Two experimental groups and two control groups were tested. In the first experimental group (hereafter referred to as the CAS group), children were asked to make two inverse transformations of each other (one adding followed by one subtracting, and *vice versa*). This gave them the opportunity to coordinate adding with subtracting and to infer the inverse relationship between addition and subtraction. In the second experimental group (hereafter referred to as the SAS group), children were asked to make two successive adding or two successive subtracting transformations. This gave them the opportunity to coordinate two identical transformations and to infer the result of such transformations. In the first control group (hereafter referred to as the NC group), children were asked to compose sets of objects that are equivalent in number with hidden sets. This control condition gave them the opportunity to produce numerical correspondences between small sets as in the experimental conditions (CAS and SAS groups). In the second control group (hereafter referred to as the NT group), children received no training of adding or subtracting. They were simply given the inverse problem twice over (as all the children in the other groups).

Our hypothesis is that it is possible to induce correct responses on the inverse problem if children are given empirical knowledge about the adding and the subtracting transformations by manipulating small sets of objects. We expect the children in the CAS group to succeed on the inverse task. But how to differentiate the effect of object manipulations and the effect of the inverse knowledge involved in CAS training? Hence the second training group in which children also generate two successive transformations but with the same directional effect on numerosity (two adding or two subtracting). If children's correct responses on the inverse problem are provoked by adding and subtracting manipulations without number representations, we expect the children in the SAS group to succeed on the inverse problem as well as the children in the CAS group. Finally, children's performance on both CAS and SAS groups have to be compared with the performance of children who have not been trained in adding and subtracting manipulations. The first control group was designed to control both the familiarization with materials and the effects of children's general involvement in producing the numerical correspondences between sets of objects (as in the CAS and SAS groups). The second control group was designed to control for the effects of two successive evaluations on the inverse problem.

None of these children had participated in the first study. They came from a middle-class population and from homes where French was the primary language. Thirty-one children were excluded because they did not reach the baseline criterion of the pre-test. A total of 44 children were therefore included in the sample used in the final analysis. The mean age of the sample was 3 years 6 months and ranged between 3 years 2 months and 4 years 1 month.

The children were shown the same wooden puppet theater and Babar dolls. In addition, 10 miniature objects each representing a Babar elephant were used in the training procedure.

The procedure for both the experimental and the control groups consisted of two phases: a pre-test in which children were given the addition, subtraction and inverse problems; and, approximately 1 week later, a post-test in which children were re-administered the addition, subtraction and inverse problems after a block of training trials for the two experimental groups and the first control group. The three problems were administered in the same way as in the first study. After the pre-test, 44 children who succeeded on both the addition $(2 + 1)$ and subtraction $(3 - 1)$ problems but failed on the inverse problem $(2 + 1 - 1)$ were assigned to four treatment groups ($n = 11$). The four groups were closely matched in mean age ($M = 3.6$, range 3.2–4.1).

Each training conditions (CAS group and SAS group) consisted of a set of trials of no fixed number. The training ended when the subject reached the criterion of success. The materials used were the same in the two training conditions. At the beginning of each training procedure, the children were shown a box containing 10 miniature objects each representing a Babar elephant. After this presentation, the experimenter (hereafter referred to as E) placed the box on the table where it was easily accessible to the children. They were asked to compose equivalent sets by adding and subtracting miniature objects in two way depending of the condition to which they were assigned.

In the CAS group, two rows of four Babar elephants were placed parallel to each other so that both rows were of the same length and those in one row were directly opposite those in the other. E designated the row nearest to him as his own and the other as the child's row and asked: "Do we have the same quantity of Babar or not?". Once the child answered appropriately, E said, "Now watch what I am going to do". Then E placed a screen between the two rows in such a way that his row was hidden. After this, E reached behind the screen and either added or subtracted one or two Babar elephants in the child's row. Note that the E's row was never modified. The added Babar elephant came from the box placed on the table and the Babar elephant removed was placed in the same box. Following this, E always asked: "Now, have we still got the same quantity of Babar? . . . . Change that to get the same quantity of Babar as me". So, the child was asked to put the Babar elephants back as they were before, either by adding or by removing some in his row. After the child's transformations, the screen was removed, and the child was allowed to verify the numerical equivalence of the two rows. A simple verbal feedback was given by E who said either "That's right" when the child performed correctly or "That's wrong" in the opposite case.

A total of four items was administered successively in the following order: addition of one Babar elephant; subtraction of one Babar elephant; simultaneous addition of two Babar elephants; simultaneous subtraction of two Babar elephants. As many trials were presented as were necessary to reach correct responses on the four items successively. Thus, children were given the opportunity (1) to produce inverse transformations by manipulating small sets of objects, (2) to verify the numerical correspondence between two sets after such transformations, and thereby (3) to infer the relationship between addition and subtraction.

The SAS group procedure was similar but with one major modification. After having placed the screen between the two rows of four Babar elephants, E transformed his row by adding or subtracting either one or successively two Babar elephants and said: "Now, have we still got the same quantity of Babar? Change that to get the same quantity of Babar as me". Here, the child was asked to carry out the same operations (and not the inverse operations) that he (or she) had previously seen. The remainder of the procedure was conducted in the same way as the CAS procedure.

A total of four items were administered successively in the following order: addition of one Babar elephant; subtraction of one Babar elephant; two successive additions of one Babar elephant; two successive subtractions of one Babar elephant. Here again, as many trials were presented as were necessary to reach correct responses on the four items in succession. In this training procedure, children were given the opportunity (1) to produce successively two identical transformations by manipulating small sets of objects, and (2) to verify the numerical correspondence between two sets after such transformations.

Only control children assigned to the NC group were administered a series of single-phase trials similar to the experimental groups. On each trial, E placed Babar elephants in a row with the number of Babars varying between two and four. Then, a screen was introduced to hide the row and the children were asked to make a row with as many Babar elephants as E had placed in the hidden row. The screen was removed as soon as the children had finished so that they could verify the numerical equivalence of the two rows. A simple verbal feedback was given by E similar to that for the two experimental groups ("That's right" or "That's wrong"). This procedure was repeated until the children made no errors. Thus, children in the NC group were given the opportunity to produce equivalence relations between two sets of the same size (as in the two experimental groups).

As expected, all children in each treatment group succeeded on the addition and subtraction problems in the post-test as well as in the pre-test (scoring was done in the same way as in the first study). Table 3 shows the number of children in each treatment group who answered correctly (and incorrectly) on the inverse problem. Both control groups did poorly compared to the experimental ones.

Of the 11 children in each experimental group (CAS and SAS groups), 10 showed increases in correct responses on the inverse problem from pre- to post-test (sign test, $P < .0044$). Only three children showed a similar drop in performance respectively in the NC control group (sign test, $P > .2482$) and 2 children in the NT control group (sign test, $P > .4795$). Correspondingly, the number of children who succeeded on the inverse problem in the post-test was significantly higher in the two experimental groups than in both NC ($X^2(1, N = 21) = 5.74$, $P < .0166$) and NT ($X^2(1, N = 21) = 8.03$, $P < .0046$) control groups.

Table 3
Number of correct and incorrect responses for inverse problem on the post-test within each treatment group

	CAS group	SAS group	NC group	NT group
Number of correct responses	10	10	3	2
Number of incorrect responses	1	1	8	9

The difference between the two control groups was not statistically significant ($X^2(1, N = 21) = .26$, $P > .6109$).

Concerning the training session administered to the CAS, SAS and NC groups, all children reached the learning criterion fixed in each procedure. The length of training never exceeded 10 min. Although the number of trials were not precisely recorded, the experimenter noted that the CAS training was more difficult for children than the SAS and NC ones. However, the maximum number of training trials never exceeded four blocks (each including four items) whatever the training group.

Since most of the children succeeded on the inverse problem in the two experimental groups (10 out of 11 in CAS and SAS groups) and since most of children failed the same problem in the two control groups (8 and 9 out of 11 in NC and NT groups), two conclusions follow. First, 3-year-olds can make correct judgments about inverse numerical transformations after a brief session of adding followed by subtracting activities. Second, the beneficial effects of CAS and SAS training only result from adding and subtracting manipulations. They can not be due to the use of numbers because numerical values were never mentioned and counting was unnecessary. Children never counted during the intervention.

It is also difficult to think that inversion was induced by the training sessions. Clearly, children in the SAS group were not trained to learn the inverse property. Even in the CAS group, this is very unlikely in view of the small quantity of training (10 min at most). Besides, as demonstrated by the results for the two control groups, the training effects are not due to familiarization with the materials, the opportunity to reason about numerical equivalencies, nor to the double evaluation on the inverse problem. So, correct responses to the inversion problem are provoked by an experiential knowledge of adding and subtracting. Of course, this is not proof that children's reasoning about numerical transformations may not be number-based. However, the results provide evidence that non-numerical object representations are a sufficient basis for children's responses.

5. Conclusion

We investigated young children's ability to reason about inverse transformations using Wynn's procedure (1992b) adapted for preschoolers from 2 to 5 years of age. We assumed that this investigation was necessary before asserting that young children have a full understanding of addition and subtraction. As pointed out by Bryant (1992), the relationship between these two operations has to be grasped in order to understand either of them properly. The concepts of addition and subtraction cannot be separated. Each, to a certain extent, is dependent upon the other because a transformation is not comprehensible *per se* but only in relation to others. It is the essential reason why Piaget and his co-workers studied all concepts as systems of reversible transformations.

Overall, the results of our both studies do not support the assumption that preschoolers understand addition and subtraction as inverse operations before the age of 4 or 5 years at least. In the first study, only six 2- and 3-year-old children out of a total of 44 correctly responded on the inverse problem. In the second study, only twelve 3-year-old children out of a total of 75 succeeded on the inverse problem on the pre-test. Therefore, young children's difficulties in grasping the inversion relation are clearly evidenced by poor performance on the "$2 - 1 + 1$" problem. This finding is consistent with training studies which have attempted to induce the conservation of number by means of the reversibility of additive and subtractive transformations (e.g., Beilin, 1965; Schnall, Alter, Swanlund, & Schweitzer, 1972; Winer, 1968; see Brainerd & Allen, 1971; Vilette, 1996 for reviews). It is not before the age of 5 years that children can infer the identity argument (*nothing adding, nothing removing*) to deduce the equivalence of two sets of objects in spite of their opposite appearance. This finding is also consistent with evidence of difficulties young children have with inverse relations in other quantitative contexts such as fractional quantities (Sophian, Garyrantes, & Chang, 1997), continuous measures (Acredolo, Adams, & Schmid, 1984) or proportional reasoning (Piaget, Grize, Szeminska, & Bang, 1968). So, this finding invites not only reconsidering preschoolers' understanding of addition and subtraction but even their ability to manipulate addition and subtraction in numerically meaningful ways.

In this respect, we have suggested that young children reasoning about numerical transformations might be object-based and not number-based. This alternative is based on the current models of quantification in infancy. Some investigators have assumed that quantitative development is guided by a domain-specific mechanism (accumulator model) to represent discrete numbers and arithmetic operations (Gallistel & Gelman, 2000; Wynn, 1998) while others have claimed that infants use general-purpose mechanisms for quantification (object-file model) without numerical representations (Simon, 1997; Uller et al., 1999). If young infants possessed an inborn mechanism for representing small number as well as procedures for calculating the exact result of simple arithmetic operations, this should be evident in preschool children. The present research do not support this assumption.

The most important evidence that young children do not use number and arithmetic principles is their responsiveness to a brief training intervention (CAS and SAS groups) in which they carry out concrete transformations by manipulating small sets of objects. Both training procedures have in common focalizing children's attention on the numerical transformations and giving them the opportunity to produce these transformations without counting. The training session just shaped children's responses in line with their experiential knowledge of adding and subtracting. Such experiential knowledge was sufficient to induce correct response on the inverse problem. This doesn't means that the children understand the inverse relationship between addition and subtraction but only that they managed to juxtapose two perceived events (without coordinating them) to make correct judgments about inverse transformations. That is what Piaget (1967) called "empirical reversibility" based on experiential knowledge and not the "true reversibility" based on logical operativity.

Our results are consistent with findings in the literature indicating that the ability to calculate simple addition and subtraction problems develops gradually throughout infancy and childhood (Cooper, 1984; Fuson, 1988; Huttenlocher et al., 1994; Starkey, 1992; Wakeley, Rivera, & Langer, 2000). Even if children are able to reason about numerical transformations long before the age of 4 or 5 years, the present research shows that they do not reason arithmetically. At a minimum, they do not fully understand addition and subtraction since the inversion principle defining the system of arithmetic is not represented. Their ability to add and subtract may be based on spatio-temporal representations of physical objects and the use of one-to-one correspondence as provided by some current conceptualizations of quantification in infancy (Simon, 1997; Uller et al., 1999) and early childhood (Huttenlocher et al., 1994; Mix et al., 2002). If so, then, children's earliest ability to add and subtract does not involve the processing of number and the use of arithmetic principles.

Of course, there is much to be learned about arithmetic development during the preschool period. Major questions concerning this period have been ignored, in particular how does the system of arithmetic emerge from the quantitative competencies of infants. The present research suggests that a non-numerical representation of quantities and transformations develops between infancy and early childhood. From this starting point, new questions arise. How does the number-based representations connect to the object-based representations? What is the role of the one-to-one mapping process in the development of both types of representation? How does the inverse relation develop gradually with addition and subtraction over the course of several years? How are they affected by factors such as the size of sets, the value of adding and subtracting, or the order of transformations? These are the new directions of research that we are pursuing in our work.

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