A Constructivist Approach to Experiential Foundations of Mathematical Concepts Revisited

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Purpose: The paper contributes to the naturalization of epistemology. It suggests tentative itineraries for the progression from elementary experiential situations to the abstraction of the concepts of unit, plurality, number, point, line, and plane. It also provides a discussion of the question of certainty in logical deduction and arithmetic. ***Approach:*** Whitehead's description of three processes involved in criticizing mathematical thinking (1956) is used to show discrepancies between a traditional epistemological stance and the constructivist approach to knowing and communication. ***Practical implications:*** Reducing basic abstract terms to experiential situations should make them easier to conceive for students. ***Key words:*** Foundations of mathematics, concept formation, conceptual semantics.

Mathematics is the science of acts
without things – and through this,
of things one can define by acts
Paul Valéry (1935, p. 811)

Preface

This is a revised version of a paper which, as an anonymous reviewer guessed, was written some time ago; in fact I wrote it in 1990/91 for the *2nd International Conference on the History and Philosophy of Science in Science Teaching*. Queen's University, Kingston, Ontario, 1992. Much has happened since then in the philosophy of mathematics, especially regarding the "naturalization" of the highly abstract concepts that were at the core of the debate on the reality of mathematical objects. As I understand it, naturalization in that discipline is the attempt to illuminate the foundations of mathematics by mathematical rather than philosophical thinking and it leads to the dismantling of the Platonist notion that mathematical objects "exist" in an absolute sense. I fervently agree with this dismantling, but my approach is on a much lower level of abstraction and focuses on how the most elementary concepts, such as unit, plurality, number, point, line, and plane could be derived from ordinary experience. I have added a postscript with references to and brief comments on publications by Brian Rotman, Penelope Maddy, and George Lakoff as samples of recent voices.

During the 1980s radical constructivism gained a certain currency in the fields of science and mathematics education. Although cognitive constructivists have occasionally referred to the intuitionist approach to the foundational problems in mathematics, no effort has so far been made to outline what a constructivist's own approach might be. This paper attempts a start in that direction. Whitehead's (1956, p. 393) description of three processes involved in criticizing mathematical thinking is used to show discrepancies between a traditional epistemological stance and the constructivist approach to knowing and communication. The bulk of the paper then suggests tentative itineraries for the progression from elementary experiential situations to the abstraction of the concepts of unit, plurality, number, point, line, and plane, whose relation to sensory–motor experience is usually ignored or distorted in mathematics instruction. There follows a discussion of the question of *certainty* in logical deduction and arithmetic.

Introduction

In a popular lecture, given at Heidelberg in 1870, Hermann von Helmholtz said that it was the relation of geometry to the theory of cognition that emboldened him to speak of geometrical subjects.[1] It is in precisely that spirit that I venture into the domain of mathematical thinking – not as a practitioner of that specific art but as a student of conceptual construction who also has an interest in education. My purpose is not to discuss mathematics as it may appear to mathematicians exercising their craft, but rather to suggest a way to think of the conceptual origin of *some* basic building blocks without which mathematics as we know it could not have developed. The constructivist approach puts in question the notion of universal conceptual "objects" and their ontological derivation and, consequently, argues against Platonic or Chomskyan innatism or any other metaphysical foundationalism.

I realize that some may consider any investigation of conceptual development an intrusion of psychologism, but even philosophers of mathematics speak of the derivation of notions: "The ideas, now in the minds of contemporary mathematicians, lie very remote from any notions which can be immediately derived by perception through the senses; unless indeed it be perception stimulated and guided by antecedent mathematical knowledge." (Whitehead 1956, p. 393)

How ideas are derived is, after all, a legitimate question for cognitive psychology, and to conjecture paths that might lead from the senses to mathematical abstractions seemed a tempting enterprise, because awareness of some experiential building blocks could help to humanize a subject that all too often seems forbidding to students. I was encouraged to pursue that question by the published evidence of wide-spread dissatisfaction with the traditional dogma among philosophers and mathematicians themselves (Lorenzen 1974; Wittenberg 1968; Lakatos 1976; Davis & Hersh 1981; Quine 1969, Tymoczko 1986a,b; Mittelstrass 1987). Since the ontological foundations of mathematics have again been put into question during recent decades, and since more and more often it is acknowledged that mathematics is the product of the human mind[2], the approach from the point of view of mental operations abstracted from experience should no longer be considered inadmissible. Indeed, once one relinquishes Plato's notion that all ideas are prefigured in every newborn's head, one cannot avoid asking how they could possibly be built up.

One might object that such an approach would be an incestuous undertaking because it obviously starts with some, albeit rudimentary, ideas of what the building blocks might be. To this I would answer that the very same pertains to all epistemological investigations, because questions about human knowledge are inevitably asked and tentatively answered by a human knower. This was inherent in Vico's (1710/1858) slogan "*verum ipsum factum*" (the true is the same as the made) and it was independently and more explicitly formulated by Kant, when he wrote "reason can grasp only what she herself has produced according to her design" (Kant 1787/1902, p. XIII). My purpose, therefore, is to isolate *possible* preliminary steps of the construction. But first I want to show my route of approach.

In his "proposals for reviving the philosophy of mathematics," Reuben Hersh (1986, p. 22) writes: "What has to be done in the philosophy of mathematics is to explicate (from the outside, as part of general human culture, rather than from the inside, within mathematical terms) what mathematicians are doing."

I am not a mathematician, and my remarks are therefore not in mathematical terms. But it should also be quite clear that I am not offering them as part of general human culture, because the radical constructivist orientation from which they spring is certainly *not* general. In my view, the part of human culture that concerns questions of knowledge and knowing, not specifically in mathematics, but in the entire experiential field, suffers from precisely the same ambivalence and hypocrisy that Hersh imputes to the philosophy of mathematics.

When Hersh writes, a few sentences after the quoted passage, that such an explication would present "the kind of truth that is obvious once it is said, but up to then was perhaps too obvious for anyone to bother saying," he manifests faith in philosophical perspicacity far greater than the perspicacity shown by our general culture in the course of the two thousand five hundred years since epistemology began. My constructivist orientation is *radical* because it proposes to cut the cognizing activity and its results loose from the traditional dependence on an assumed ontology. It is an attempt to do without the notion of truth as a representation of an experiencer-independent reality, material or metaphysical (cf. Glasersfeld 1989). Hence, the approach I am expounding here may upset not only "Platonist" mathematicians but all who are philosophically or emotionally tied to some form of realism. My intention, however, is simply to contribute to a discussion that is still wide open.

Mathematics and communication

Concerning criticism in mathematical matters, Whitehead (1956, p. 395) explained that "…there are always three processes to be kept perfectly distinct in our minds." The way Whitehead has formulated and explicated these three processes provides a good basis for laying out some features of the constructivist approach.[3] We must first scan the purely mathematical reasoning to make sure that there are no mere slips in it – no casual illogicalities due to mental failure. Any mathematician knows from bitter experience that in first elaborating a train of reasoning, it is very easy to commit a slight error which yet makes all the difference. But when a piece of mathematics has been revised, and has been before the expert world for some time, the chance of casual error is almost negligible. It seems difficult not to agree with this statement, since it describes a checking procedure with which we are quite familiar (e.g. when someone is hanging a picture, drawing a map, writing a computer program, and so forth). On second thought, however, we may notice that Whitehead refers to an expert's check of another person's "purely mathematical reasoning," which he then calls "a piece of mathematics." This may prompt us to ask where the expert finds the entity that is to be checked. If it was a piece of mathematical *reasoning*, it must have been generated by someone's thinking. Since we cannot read minds, access to another's thinking or reasoning requires an act of communication. In other words, before a piece of reasoning can be checked, it must be formulated in some language or symbols that are known to both the author and the expert who is to do the checking. (From the constructivist point of view, communication is itself problematic, and I shall deal with it in the context of Whitehead's second process, where it is even more relevant.)

For formalists, there should not be much of a communication problem. They take the presented symbols as they find them, and check whether or not they have been combined according to generally accepted rules. Formalists are concerned with syntax, not with conceptual semantics. The author's acts of *reasoning* would not be questioned, because – although formalists do not usually say this explicitly – from their perspective, symbols have to be perceived but need no *interpretation*, and the correctness or error of a mathematical expression depends exclusively on its formal compliance with the rules of the chosen symbol system.

That this is not a very satisfactory approach to questions of the mathematical underground, has been remarked by many critics during recent decades. Hersh (1986, p. 19) has expressed the objection in a very general way: "Symbols are used as aids to thinking just as musical scores are used as aids to music. The music comes first, the score comes later." That the score ought to conform to the rules of the scoring system, therefore, is simply a precondition to any judgment concerning what the score might represent on the conceptual level. Mathematical symbols, of course, are far more complex and layered than musical notation and some of the conceptual "music" they are intended to signify presumably arises on the higher levels of symbolization. In what follows, however, I want to focus on the very lowest level, the level on which non-mathematical experiences provide material for the abstraction of the most elementary mathematical building blocks.

Having discussed the possibility of "mere slips," Whitehead turns to the starting-points of mathematical reasoning.

He writes: "The next process is to make quite certain of all the abstract conditions which have been presupposed to hold. This is the determination of the abstract premises from which the mathematical reasoning proceeds. This is a matter of considerable difficulty. In the past quite remarkable oversights have been made, and have been accepted by generations of the greatest mathematicians. The chief danger is that of oversight, namely, tacitly to introduce some condition, which it is natural for us to presuppose, but which in fact need not always be holding." (Whitehead 1956, pp. 395–396).

The presupposition of unwarranted conditions is something radical constructivism claims to have unearthed in several areas that have no obvious connection with mathematics. One area, however, that *is* relevant for the discussion of thinking and, as philosophers of mathematics have found, is indispensable for any critique of mathematical reasoning, is the kind of social interaction we call linguistic or symbolic communication (cf. Davis & Hersh 1981; Tymoczko 1986a).

Where communication is concerned, we habitually – and hence mostly tacitly – operate on the presupposition that others who use language or symbols that we readily recognize as such, are using them with the same meanings that *we* have come to attribute to them. This assumption is made habitually, because without it, most if not all of our everyday linguistic and symbolic interactions would be futile. Indeed, we are constantly reinforced to assume that our meanings are shared by others, because by and large our ordinary communicatory interactions work remarkably well. But the bulk of our ordinary communicating is about two experiential areas. Either it concerns sensory–motor objects, where perceptual feedback helps us to avoid gross misinterpretation; or it concerns emotions, and in the emotional sphere, where meanings are notoriously vague, the margin for interpretation is so wide that we are rarely compelled to consider feedback that upsets the pleasant generic feeling of understanding or being understood.

In contrast, when we are communicating within the relatively systematized domain of a science, we are dealing to a large extent with abstract concepts and relations. In this area, such feedback as we do receive about the other's *reasoning* usually springs from inferences we draw from the technical context, *as we see it*, or from the other's actions, *as we observe them.* Whether the respective concepts are actually the same, cannot be ascertained, but insofar as scientific training imposes conventions of thinking, a certain degree of compatibility can be assumed.

Social adaptation and compatibility

For Whitehead, almost twenty years after *Principia Mathematica*, there was no doubt that there was a fixed set of logical rules that formed the solid basis of mathematics. And the rules of logic were taken to be *a priori* and therefore not only unquestionable but also inevitably inherent in every thinker's rational procedures.

Like the methodology of Ceccato's Italian operationalist school[4], radical constructivism is an attempt to do without the assumption of *a priori* categories or rules. Categories are seen as the results of mental construction (as in Piaget's theory, in the case of space, time, and causality[5]); the sort of rules that make possible the repetition and checking of *logical* procedures are based on the coordination of specific mental operations with specific symbols. This coordination has to be accomplished by every single thinking subject, and the "intersubjectivity" that makes possible the communication of logical procedures can be achieved only through the individuals' social interaction and mutual adaptation of their subjective coordinations (of mental operations and symbols). This is what Maturana (1980) has called a "consensual domain" generated by the "coordination of the coordinations of actions." In this view, then, the *meaning* of both natural language and mathematical symbols is not a matter of "reference" in terms of independently existing entities, but rather of subjective mental operations which, in the course of social interaction in experiential situations, achieve a modicum of intersubjective compatibility.

After his exposition of the three critical processes, Whitehead made the very important general remark: "…the trouble is not with what the author does say, but with what he does not say. Also it is not with what he knows he has assumed, but with what he has unconsciously assumed" (p. 396). This becomes particularly relevant when he discusses the third process.

"This third process of criticism is that of verifying that our abstract postulates hold for the particular case in question. It is in respect to this process of verification for the particular case that all the trouble arises. In some simple instances, such as the counting of forty apples, we can with a little care arrive at practical certainty. But in general, with more complex instances, complete certainty is unattainable. Volumes, libraries of volumes, have been written on the subject. It is the battleground of rival philosophers. There are two distinct questions involved. There are particular definite things observed, and we have to make sure that the relations between these things really do obey certain definite exact abstract conditions. There is great room for error here. The exact observational methods of science are all contrivances for limiting these erroneous conclusions as to direct matters of fact." (p. 396)

Whitehead then discusses the problem of ascertaining that an experiential object is actually of "the same sort," so that one can ascribe to it a condition that was abstracted from a prior sample. "The theory of Induction," he says, "is the despair of philosophy – and yet all our activities are based upon it."

The problem of induction which, as Whitehead defines it here, is the question whether one could justify the *generalization* of an idea that was abstracted from a particular sample of experiences, concerns the entire domain of science, not specifically mathematics. Here, I therefore merely emphasize that, from the constructivist point of view, "particular definite things observed" and "direct matters of fact" are generated in ways that differ from the conventional realist account implied by this passage from Whitehead. The "practical certainty" in the case of the forty apples, however, falls within the realm of experiences that are crucial to the thesis I want to develop, because according to it, the certainty does not spring from the counted "objective" things, but from the mental operations of the counter (cf. for instance Piaget 1977, p. 71).

I have mentioned that, from the constructivist point of view, there are problems in discussing an individual's mathematical reasoning, because, in order to be discussed, criticized, or assessed in any way, this reasoning must first be uttered or written and then *interpreted* by the critics or evaluators. To this

I now want to add that whatever Whitehead intended by the expression "direct matters of fact" is not something that is an unquestionable *given,* but rather the result of an individual's *interpretation* of experience which, to that individual, *appears to be compatible* with the interpretations of other individuals. Compatibility – and this cannot be stressed too much – is not the same as identity. The distinction is important, because it alters the notion of "understanding," of "shared" ideas and conceptual structures, and thus also the notion of "accepting a proof." When I agree with what another says or does, it means no less, but also no more, than that I interpret the statement or the action in a way that manifests no discrepancies from what I might say or do under the given circumstances *as I see them*. This leaves room for uncounted discrepancies that did not happen to surface; and we all know how often, after a feeling of agreement, a feeling of harmony and complete understanding, some further interaction brings out a discrepancy that had remained hidden in all that went before.

Tymoczko (1986b, p. 48) maintains that the type of criticism that is necessary for the development of proofs is essentially social: "Without criticism, proof-ideas cannot develop into proofs." In other words, when some reasoning, presented in speech or writing, is claimed to constitute a proof, this communication has to be interpreted and criticized by others. Then the critique has to be interpreted and answered by the author, and only when the critics interpretation of the answer is judged adequate by them, will the presented reasoning be promoted to the status of proof. No matter how many iterations this procedure might go to, it is clear that it can never guarantee the identity of concepts and conceptual relations used by the different thinkers involved. It can at best lead to apparent compatibility – and this, of course, is quite sufficient for the *practice* of mathematics (or any other domain of human cooperation).

But if compatibility is all that can be achieved, we cannot found mathematics in an ontological realm of ideas presumed to exist independently of any thinking subject. Yet Lakatos (1976/1986, p. 44) was probably right when he remarked: "It will take more than the paradoxes and Gödel's results to prompt philosophers to take the empirical aspects of mathematics seriously…" From the outside, however, witnessing the crumbling of what were considered the foundations of mathematics, it seems reasonable to ask what the most elementary building blocks could be that might serve as a basis for the constitution of the mysterious structures or "objects" that mathematics develops. To pursue that quest requires an empirical investigation, where "empirical" has its original meaning and refers to *experience*. But clearly it will not be sensory experience that matters, but the experience of mental operations. As Hersh (1986, p. 22) put it: "mathematics deals with ideas. Not pencil marks or chalk marks, not physical triangles or physical sets, but ideas (which may be represented or suggested by physical objects)."

If we do not want to believe that ideas are innate or God-given, but the result of subjective thinkers' conceptual activity, we have to devise a model of how elementary mathematical ideas could be constructed – and such a model will be plausible only if the raw material it uses is itself not mathematical.

To begin at the beginning

To have the idea of counting, one needs the experience of handling coins or blocks or pebbles. To have the idea of angle, one needs the experience of drawing straight lines that cross, on paper or in a sand box. (Hersh 1986, p. 24)

Unless we are able to conceive of something that is unitary, in the sense that we distinguish it as a discrete item in our experiential field and are able to "recognize" other items *like* it, we cannot have a plurality. If we have no plurality, we have no occasion to count – and if we did not count, it is unlikely that we should ever have arithmetic and mathematics, because without counting there would be no numbers. Hence, if we want to get some inkling as to how arithmetic arises, we may have to begin with the concepts of unit and plurality, as well as the activity of counting which generates the concept of number. The ensuing development was described by Paul Lorenzen (1974, p. 199):

"The foundation of arithmetic is the prearithmetical praxis: the use of counting-signs (e.g. |, ||, |||, ||||, …) in counting collections (heaps, herds, groups, complexes, …); using counting-signs rather than the collections themselves, to make comparisons of amounts; adding and subtracting counting-signs (instead of certain operations with the collections)."

Rotman (1987, p. 8) uses the same notation of counting signs, but both he and Lorenzen seemed to take the concepts of unit and collection for granted. Yet, the "prearithmetical praxis" obviously does not begin with counting-signs. Before any collections can be counted and coordinated with signs that can be used in their stead, experiential items must be gathered in collections such as heaps or herds. To do this, we must distinguish more than one experiential item, i.e., a *plurality*, such that each of the individual items satisfies whatever conditions govern membership in the particular heap, herd, or collection we want to form.

The first task, then, is the distinction of individually discrete "things" in our experiential field. To normal adult humans, who are experienced managers of a more or less familiar environment, it may seem absurd to suggest that the segmentation of their experiential world into discrete things should not be an ontological given. But even the most orthodox epistemologists, at least since the days of John Locke, have discarded this commonsense notion. Many modern scientists, e.g., Mach (1910/1970, p. 42) and Bridgman (1961, p. 46), explicitly stated this, and Albert Einstein (1954, p. 291) formulated one of the simplest, uncompromising descriptions of how we come to furnish our world with discrete things:

"I believe that the first step in the setting of a 'real external world' is the formation of the concept of bodily objects and of bodily objects of various kinds. Out of the multitude of our sense experiences we take, mentally and arbitrarily, certain repeatedly occurring complexes of sense impressions (partly in conjunction with sense impressions which are interpreted as signs for sense experiences of others), and we correlate to them a concept – the concept of the bodily object. Considered logically this concept is not identical with the totality of sense impressions referred to; but it is a free creation of the human (or animal) mind."

Piaget (1937) provided a minute analysis of how object concepts might be constructed by the very young child, and from Edmund Husserl we have a suggestion that is particularly relevant to the present context. In his *Philosophie der Arithmetik* (1887/1970), Husserl proposed that the mental operation that unites different sense impressions into the concept of a "thing" is similar to the operation that unites abstract units into the concept of a number (p.157–168). I accept this hypothesis, but want to point out that, in order to have several units that can be united to form a number, one must have a concept of plurality. I therefore want to unravel the steps involved in the initial development and fill in some of the details that seem necessary.

Compounds of sensory impressions presumably acquire their first stability, as Einstein suggested, through repetition. Brouwer (1949) proposed that the perceiving subject's self-directed attention "performs identifications of different sensations and of different complexes of sensations, and in this way, in a dawning atmosphere of forethought, creates *iterative complexes of sensations*" (p.1235; emphasis in the original). The stability of such a sensory compound manifests itself in the subject's ability to "recognize" it when it is produced again. Children clearly show this in the early stages of language acquisition. Once they have isolated a group of sense impressions and have associated them with, say, the word "cup," they may toddle to the kitchen table and, pointing with their finger, say "cup," then point to another and say "cup," and repeat this procedure for every cup they happen to perceive on the table. Psycholinguists call this phenomenon "labeling" and it provides good evidence that some kind of structure has been formed which allows children to utter the associated word whenever their perceptual mechanisms produce sensory signals that can be fitted into that particular structure.

The abstraction of plurality

However, to say that a child that does this has a concept of *plurality,* would be reading too much into the episode. In fact, it usually takes at least a month or two before the child will use the plural "cups" instead of acknowledging individual cups singly. This delay cannot be explained by the simple fact that the child has to learn a different word (i.e., the plural form), because in order to use the new word, the child also has to learn to isolate a different experiential situation. The plural "cups" must be associated with a perceptual situation that contains *more than one* cup – and this "more than one" is *not* a perceptual fact. The inference that more than one cup is on a table is not based on the sensory impressions isolated as cups, but on the *awareness* that one has repeated the same operations of isolating and recognizing within certain boundaries of space and time. The conception of a plurality, therefore, is the result of a repetition of mental operations that accompany the sensory impressions but are themselves not sensory. And the basis on which the conceptual structure called "set" can be created is laid only when, in a further step of abstraction, the operations that generated a plurality are seen as an operational pattern *without* considering the sensory items that constituted the collection[6].

Consequently, the notion of a plurality of things, for which language supplies the plural form of the things' name, should not be considered a mathematical concept. Though it results from mental operations, it is tied to sensory experience and does not require an abstract concept of unit. Indeed, Husserl makes clear that the ordinary meaning of the word "one" (used in opposition to a plurality, as in "more than one") must be distinguished from the *numerical* concept of "one," which designates an abstract unit (Husserl 1887/1970, pp. 128ff).

That the abstract units required in arithmetic are not quite the same as the units constituted by the discrete objects in our experiential world, was indicated also by Frege (1884/1974, p. 58), when he said that the things we number must be distinguishable, whereas the units of arithmetic are not, because they are conceptual and have to be identical in every instance.

Brouwer suggested that the fact that attention can be directed, enables the mind to produce the complexes of sensory elements that we perceive as "things," and that this ability derives from the inherent character of consciousness which, in his view, was not a steady state but an oscillating function (Brouwer 1949, p. 1235). A similar idea was proposed quite independently, by Silvio Ceccato (1966), who posited a pulsating attentional mechanism that generates patterns of focused and unfocused pulses which can then be used to segment sensory material into iterable conceptual structures. I have further developed this approach in a hypothetical model of the construction of the concepts of unit and number (Glasersfeld 1981). The model leans on Piaget's notion of reflective abstraction[7] and offers at least an hypothetical skeleton of mental operations that lead from the inception of discrete sensory things to pluralities, collections, arithmetic units, set, and number. Here, however, I want to concentrate, not on the mechanisms of abstraction, but on experiential elements that may serve as its raw material. In what follows, I shall make the case that counting provides the most plausible basis for the abstraction of the concept of "numerosity" (the cardinal aspect of number).

Counting and number

If we want to agree with what Hersh says in the quotation I have placed at the beginning of this section, it will be necessary to make quite clear what we mean by "counting." The word has been used for diverse manifestations that range from a toddler's meaningless recitation of a few number words to the function of a gadget that indicates radioactivity. From the constructivist point of view, counting is a very specific, complex activity (cf. Steffe et al. 1983). According to our definition, it has three components: A conventional number word sequence, a plurality of unitary sensory items (perceived or visualized), and the one-to-one coordination of successive number words and the items in the collection.

Imagine an ordinary, nonphilosophical observer watching a mason who utters the standard sequence of number words as he points to the bricks lying beside him. If "eighteen" turns out to be the last number word to which a brick could be coordinated, it should be quite clear to the observer *why* the mason might now announce that there are eighteen bricks in the counted collection. If the observer has been attentive and neither missed, nor found fault with, any of the steps in the mason's procedure, she herself has indeed come to the same result. Nevertheless, if one asked either of them how they *know* that there are eighteen bricks, the chances are that they would both answer: "Well, I just counted them!" It is unlikely that either of them would explain in sufficient detail the procedure that was carried out and why the last number word used could be taken to indicate the numerosity of the collection.

We have all learned to count in early childhood, and we take for granted that the last number word of a count tells us *how many* items there were involved. We are, indeed, so accustomed to this, that we do not consciously think, whenever we hear or read a number word, that it entails a count of as many discrete units as there are number words in the conventional sequence leading up to it from "one." Yet, if we did not know this in some way, the number word could have no meaning for us. The fact is that number words have become symbols for us, and as such they symbolize the counting procedure that leads up to them, *without our having to carry out* that procedure or even having to think of it.[8] Even in the case of numbers that are higher than we could ever actually reach by counting, the tacit knowledge that there is a procedure by means of which, theoretically, we could reach them, constitutes the first (but by no means the only) characteristic of *number* as an abstract concept.

In short, I submit that the three elementary concepts of arithmetic – unit, set, and number – are abstractions, not from physical objects or other sensory material, but from mental operations that thinking subjects must carry out themselves. At the beginning, ontogenetically speaking, these operations develop as corollaries of actions which, in order to be performed, require sensory–motor material. This material need not be the same for all thinking subjects, it merely provides the occasion. However, once patterns of mental operations have been abstracted, they become *mathematical* concepts through association with symbols that can "point"[9] to them without invoking their actual execution.

The concept of unit is abstracted from the perceptual operation of combining various sense impressions to form a "thing"; the concept of set is derived by abstracting the plurality of abstract units from a collection of things (i.e. considering an experientially bounded plurality but not the sensory items that were used to generate it); the concept of number arises when number words or numerals have become symbols that tacitly point to a possible count that leads up to them.

I want to emphasize that this analysis concerns the basic inception of the concepts, and that in the vast domain of mathematics each of them can be indefinitely enriched by the addition of further abstractions. Besides, there is the whole area of geometry to which I now want to turn.

From action to abstraction

The understanding is a wholly active power of the human being; all its ideas and concepts are but its creation,… External things are only occasions that cause the working of the understanding … the product of its action are ideas and concepts.
Kant (1787/1902, Vol.VII, p. 71)

In the beginning, geometry is usually presented as a matter of points, lines, and planes. In ordinary language, we have no difficulty in finding experiential objects to which we can apply these words. Most of these objects, however, do not satisfy the mathematician's requirements. Hence there are mathematical definitions, or rather, expressions that purport to serve that purpose. But even mathematicians themselves are not always pleased with them and therefore add more technical formulations, based not on experience but on a specific mathematical frame of reference, such as a system of orthogonal coordinates. Thus my *Mathematical Dictionary* (James & James 1959, p. 274) has the following entry for "point":

(1) An element of geometry which has position but no extension.

(2) An element of geometry defined by its coordinates, such as the point (1,3).

Although it is a long time ago, I can still remember our teacher, before we had heard anything about coordinates, making a small mark on the blackboard and saying, as he turned to us: "This is a point." Then he hesitated for a moment, looked back at the mark, and added: "Of course, a geometrical point has no extension." This left us wondering about grains of sand, specks of dust, and other smallest items, but we remained perplexed because all of them still had *some* extension. After a few days the perplexity was forgotten. We had learned to make points with our pencils, and all that mattered was that they were not noticeably wider than the lines we drew.[10] The obstacle here is that, logically, it is impossible to move smoothly from small and smaller to "no extension," and yet hold on to a something that could be discriminated; and it is equally impossible to move from few and fewer to "none," and yet hold on to the notion of plurality. Hence, what is needed is another approach. When we first meet "zero," we can see it as the number word to use when all countable items have been taken away. And there is an analogous approach to "point" in a visual experience that most will have had in one way or another. Imagine, for instance, sailing away, on a perfectly smooth lake, from a small floating object, say, a bottle. It gets smaller and smaller, and suddenly you cannot see it any longer, though you are still looking at the point where it was. The bottle is gone, and it would seem more adequate to identify the point with the focus of your attention. This gets rid of the problem of size, because it is never the focus of attention that has a size, but only the *things* one is focusing on.[11]

Hence I propose to think of "point" as the very center of the area in the focus of attention. In the visual field, then, it would be the center of an item we are focusing on. If that item is so small that we cannot distinguish a center from the circumference, we have an item that can *represent* our concept of point, but it is not itself a point, because we can still imagine that it *has* a center, even if we cannot see it. But then this center turns into the vanishing point, a conceptual construct that derives from movement and attention.

Another way to approach the concept of point was suggested by Ceccato in conversations we had around 1950: Think of a form of cheese, he said, and the way one cuts it by pulling a wire through it. (This, of course, was in Italy, where large forms of cheese are always cut in this way.). The first cut, say a vertical one, gives you a plane. If you cut again vertically, intersecting the first plane, the two cuts give you a vertical line. And if now you cut horizontally, the intersection of the three cuts gives you a point. What you have to focus on, of course, is not the wire, nor the space it leaves, but the movements, because in movements we feel direction but no lateral extension.[12]

To my mind, both these approaches are more adequate than merely *saying* that a point has no extension. They come closer to describing what one can *do* to arrive at the concept that has no sensory instantiation.

In the case of "line," there is a reputable precedent. In his *Critique of pure reason,* Kant (1787/1902, p. 138) says, in order to experience a line, one must *draw* it. But he said it in German, and translation, as so often, obscures the original meaning. The English verb "to draw" is used indiscriminately for producing images with a pencil and for what horses do to a cart. In German the two activities require different verbs. Kant used *ziehen*, which means "to drag" or "to pull" and does not refer to a graphic activity except in the case of lines. Since he spaced the word (to emphasize it), he had in mind the physical motion.[13]

In the same vein, Brunschvicg (1912/1981, p. 503) says of the straight line: "The elementary operation that is to furnish the simplest image is the stroke (*trait*). The hand places itself somewhere, it stops somewhere; from the point of departure to the point of arrival, the mind has not become aware of a division or a change in the movement accomplished by the hand. Hence there is no reason to suspect that the path linking the two points might not be a uniform and unique line, a straight segment that could serve to measure the distance. In fact, we know by what roundabout way geometry was led to question the evidence that seems to support the uniqueness of the straight line between two points; and we understand that it was the ease and certainty (of the act) that enabled the mind to cling to what is given by intuition" (my translation).

Intuition, here, I suggest is precisely what Kant meant when he said *Anschauung,* i.e., the view of an experience upon which we are reflecting. The line, then, is a reflective abstraction from a uniform movement we make. To this I add, that this movement need not be that of a hand or other visible object, but it can be the movement of our attention in whatever field we happen to be considering. And the straightness of an object can, in practice, be checked by shifting the focus of attention in uniform motion, as cabinet makers do when they look along the edge of a board to check that it is not warped.

A second dimension

Having found an experiential basis for points and lines, we may follow the definition of "plane" in James & James (1959, p. 273): "A surface such that a straight line joining any two of its points lies entirely in the surface." Implicit in this criterion is the requirement to look in at least two directions, which is equivalent to *drawing* more than one line. Hence the axiom that a plane is defined by three points. But each of these defining elements itself involves only one direction and therefore provides no immediate experience of the two-dimensionality of surfaces or planes.

It has also been suggested that a plane can be constructed by moving a line sideways. This is interesting, because in order to follow this movement, the focus of attention has to be widened to cover at least a certain stretch of the line. A practical equivalent would be to move one's hand on a surface, and feeling no change either in the tactile pressure of fingers and palm or in the direction of movement. In both cases there is an expansion of the focus of attention in order to monitor more than one point. And this expansion is the opposite of the shrinking in the example of the bottle on the lake.

Confrey (1990) proposed that the way an object increases or decreases in size, as we move towards or away from it, provides an experiential basis for both exponential change and geometric similarity. I fully accept this idea and want to stress that, in the present context, its most relevant aspect is the expansion, respectively contraction, of the area covered by our attention. This movement provides experiential situations from which, in the expanding direction, the two-dimensionality of the plane can be abstracted, whereas the shrinking direction may lead to the abstraction of a concept of point.

These brief suggestions are the merest beginning of an analysis of the conceptual foundations of geometry. Other experiential situations can be found that may serve as raw material for the abstraction of the traditional "basic elements," and none is unique in the sense that it could not be replaced by another. But as with the concepts of unit, plurality, and number, I believe that the minute analysis of elementary actions and operations that might occasion their abstraction is a direction of research that is well worth pursuing. If students, at the time when geometry is introduced, were offered experiences of this kind, they might come to understand that the lines drawn on paper and the physical models of bodies they are shown are merely occasions for mental operations that have to be actively carried out in order to abstract the basic concepts of geometry.

A source of certainty

Mathematics, like theology and all free creations of the Mind, obeys the inexorable laws of the imaginary. – Gian-Carlo Rota (1980, p. XVIII)

In the preceding sections I have argued that common non-mathematical activities, such as isolating objects in the visual or tactual field, coordinating operations while they are being carried out, and generating a line by a continuous uniform movement, are the experiential raw material that provides the thinking subject with opportunities to abstract elementary mathematical concepts. If one accepts this view, one is faced with the puzzling question how such obviously fallible actions can lead to the certainty that mathematical reasoning seems to afford.

The puzzle is not unlike the one that arises if we write the traditional textbook syllogism with a first premise that we assume to be false – for instance, "All men are immortal." If we proceed with "Socrates is a man," the conclusion that Socrates is immortal will be just as certain and *logically* "true" as the opposite conclusion, which we get when we start with the more plausible first premise that asserts the mortality of all humans.

This puzzle disappears if it is made clear that the premises of a syllogism must be considered as *hypotheses* and should be preceded by "if." Their factual relation to the experiential world is irrelevant for the formal functioning of logic. Considering them to be "as though" propositions, makes sure that, for the time being and during the subsequent steps of the procedure, one is not going to question them. The steps of that procedure are, on the one hand, the specific mental operations designated by terms such as "all," "some," "no," "is a," "then," etc., *and,* on the other hand, the operation of combining the two premises. Assuming that these operations are carried out in the customary way, the certainty of the conclusion springs from the fact that the situations specified by the premises are *posited* and, therefore, not to be questioned during the course of the procedure.[14]

Viewed in this way, the syllogism also becomes immune to a criticism which, I believe, was first brought up by John Stuart Mill (1843, A system of logic, III.2). Because he was mainly concerned with the logical uncertainty of inductive generalization, he remarked that, in order to be justified in saying that *all* men are mortal, one should have examined all members of the class called "man." If Socrates is rightly considered a member of this class, one must have come across him during the examination and one does not need the conclusion, to know that he is mortal. If, however, one has not examined Socrates, then either the second premise or the conclusion is false. – When an "if" is placed before the premises, this voids the above argument because it obviates the examination of all members of the class involved; and what the conclusion affirms is made unquestionable, because this now derives from what was hypothetically posited and is therefore not dependent on an examination of actual experiences.

The hypothetical trick

Yet there remains a question. How do we come to feel certain that the conclusion tells us something that was contained in the premises? This goes to the core of the deductive procedure and is crucial for my thesis. If the major premise is of the form "all X are B," it implies that there is a *bounded collection* of Xs, either experiential or hypothesized. But collections are the result of mental operations, and to form a collection, we clearly must first have formed a plurality. In turn, to form a plurality we need to conceive of discrete unitary items that have some attribute, let us say A, in common. If we now consider a discrete unitary item and find that it has the attribute A, we may say: "Ah, here is another X" (and we may proceed to examine whether it, too, is B, and thus fits our hypothesis). But note that, to say "another," we must *remember* that at some earlier moment we attributed A to some item(s). Similarly, to conclude in the syllogism that Socrates is mortal, we must *remember* that we previously formed a collection called "man" and attributed mortality to it, because this is the basis on which we now feel certain that, if Socrates can be considered a member of that collection, he must be mortal.

I believe it was Euler, who first used circles to indicate the bounded collections involved in propositions such as "Some A are B" and "all A are B." (Euler 1770, letters 102–108, 14 February to 7 March, 1761). And he went on to show how the syllogistic procedure could be visualized in this manner. The mystery of logic, purported to be so difficult to approach, he says, immediately strikes the eye if one uses these figures (Euler 1770, letter 103). And he is right. Two concentric circles do convey the notion of containment with great force – but at the moment of perceiving this symbolic containment, one still has to remember that the outer circle is intended to stand for the major premise and the inner circle for the minor. The circles help to make palpable the relation but not *what* is being related. Hence, no matter how we look at it, the judgment of *certainty* involves faith in the flawless functioning of the particular kind of memory we use in carrying out the syllogistic procedure.

The "certainty" in arithmetic is analogous in that it, too, depends on mental operations and not on a fit with the experiential world. If someone tells me that there are seven oranges in the kitchen, three on the table and four on the windowsill, I do not have to accept this as an unquestionable "truth" – even if I believe that he or she is using the number words the way I myself would use them. There might have been an error, either in recognizing oranges or in counting. But I cannot question the simple statement that $3 + 4 = 7$. Unpacked, this statement means: You count a collection and come to "3," then you count another collection and come to "4"; now you can consider the two collections as one collection, and if you count it, you will come to "7." Provided one's procedure follows the standard number word sequence and coordinates its terms to countable items in the standard fashion, one is going to arrive at the standard result every time. Because the operations involved have become habitual and we are not aware of carrying them out, we get the impression that there is something preordained about their results, something that could not be otherwise. As Spencer Brown said about the existence of the universe: "It comes through a very clever trick. It depends on an elaborate procedure for forgetting just what it was we did to make it how we find it" (Keys 1972, p. 31).

Hence, there is the involvement of memory. In order to know (when you have finished counting the composite collection and came to the result "seven") that 7 is the sum of a collection of 3 and a collection of 4, you must *remember* that you counted these two collections before you combined them. As in the case of the syllogism, a graphic representation of the procedure, say "prearithmetical signs" like those Lorenzen suggested for counting, may help to visualize the procedure – but in order to tie the graphic signs to the process of addition, you have to remember the meaning that was attributed to them at the outset.

This leads to the conclusion that one can do neither logic nor mathematics without *doing* things which, themselves are not specifically mathematical. Depending on the kind of mathematical result aimed at, there will be activities such as isolating discrete perceptual or visualized items, moving a limb or the focus of attention, attributing meanings to signs or symbols, considering explicitly or implicitly limited contexts, and remembering the conceptual commitments that have been made during these mental operations. The certainty of the results, then, springs on the one hand from the fact that one operates in a hypothetical mode and therefore obliges oneself not to question what one has hypothesized; and on the other hand, on implicit faith in one's memory of meanings attributed, of operations carried out, and of the results they produced.[15]

Conclusion

If what I have outlined is a viable approach, there are several things that will have to be considered when the foundations of mathematics are discussed. First, they cannot be discussed without the use of language. From the radical constructivist point of view, there is no neat distinction between private and public language, because all meaning of signs and symbols, including the linguistic kind, is built up by individuals on the basis of their own subjective experiences of isolating objects, events, and the relations among them. No doubt everyone's meanings are modified and adapted in the course of interaction with other speakers, but the result of such adaptation is *compatibility*, not identity. And the compatibility achieved is relative to the particular interactions an individual has participated in. The expression "shared meaning" is a deceptive fiction because sameness can never be ascertained.

Second, some (and perhaps all) of the indispensable elements in mathematical thinking are conceptual constructs that were abstracted from operations carried out with sensory material, operations that are involved in segmenting and ordering experience long before we enter into the realm of mathematics.[16]

It seems to me that these are two good reasons for considering mathematics to be an "empirical" enterprise. Lakatos and others (e.g., Davis & Hersh 1981, Tymoczko 1986a) have called it "quasi-empirical" because it does not directly deal with physical objects. To a radical constructivist, however, physical objects, too, are conceptual constructs abstracted from a way of experiencing that imposes structure on an essentially amorphous sensory manifold. Mathematics deals with constructs that no longer contain sensory or motor material because they are abstracted from mental operations carried out with that material; but this does not make them any less experiential – and "empirical," after all, is but another word for "experiential." The poet/mathematician Paul Valéry has said this with uncommon elegance in the epigraph I placed at the beginning.

Postscript

Penelope Maddy (1997, p. 233) proposed an approach "that turns away from metaphysics and towards mathematics." What concerns her is not the justification or ontological reality of the relevant concepts but how mathematicians have formed them. The formation she describes takes place on a level of abstraction beyond the very first one that uses sensory experiences as raw material (which is the topic of my paper). Set theory has been so successful, she claims, because it has created its own conceptual ontology that does not require the Platonic ontology philosophers are mainly unwilling to relinquish. But as Quine (1969, p. 83) put it: "The old epistemology aspired to contain, in a sense, natural science; it would construct it somehow from sense data. Epistemology on its new setting, conversely, is contained in natural science, as a chapter of psychology." As I understand it, Maddy's suggestion is that foundations of mathematics have to be found within set theory.

Encouraged by the popular success of *Metaphors we live by* (Lakoff & Johnson 1980), George Lakoff (Lakoff & Núñez 2000) embarked on the exploration of the metaphors that, he believes, form the source of mathematics. The meaning of mathematical terms, he claims, consists in conceptual metaphors. Each conceptual metaphor is "a unidirectional mapping from entities in one conceptual domain to corresponding entities in another conceptual domain" (Lakoff & Núñez 2000, p. 42). From my point of view, this is a highly misleading statement. In my world, metaphors can be defined as the attempt to transfer a property assumed to be characteristic of one type of experiential item to an item that is usually considered not to have that property. (E.g. if I say: "My brother in law is a gorilla," it is up to you to discover from the context which of a gorilla's properties I am attributing to my brother in law. But this is not the only kind of metaphor Lakoff has in mind. He also wants to call metaphor any relational pattern that has been abstracted from an experiential situation. This happens to be exactly what Peirce called an "abstractive observation"; and Piaget some thirty years later called it "reflective abstraction," showing how patterns of physical action could lead to patterns of mental operating. Piaget is mentioned in a marginal context only, Peirce not at all, and, what is even more astonishing, there is no reference to Brouwer, who explicitly linked the origin of number to an experiential situation (cf. Section "To begin at the beginning" above). Lakoff & Núñez's *Where mathematics comes from* (2000) suggests a great number of applications of his theory of metaphor to mathematical concepts. The usefulness of this move will have to be judged by mathematicians. However, Lakoff says practically nothing about the elementary concepts which I deal with in my article. He takes discrete countable things and pluralities for granted and his account of counting makes no mention of a conventional number-word sequence, without which, I would claim, no concept of number can be formed.

Brian Rotman (2000, p. 38), judging by what he wrote in *Mathematics as sign*, would agree with what I wrote in Section 5, above. "Numbers," he says, "appear as soon as there is a subject who counts." He justifies this in the preceding paragraph: "However possible it is for them (i.e. whole numbers) to be individually instantiated, exemplified, ostensibly indicated in particular, physically present, pluralities such as piles of stones, collection of marks, fingers, and so on, numbers do not arise, nor can they be characterized, as single entities in isolation from one another: they form an ordered sequence, a *progression*. It seems impossible to imagine what it means for "things" to be the elements of this progression except in terms of their production through the process of counting." As a general maxim he states that "A mathematical assertion is a *production*, a foretelling of the result of performing certain actions upon signs … Thus, for example, the assertion '2 + 3 = 3 + 2' predicts that if the Subject concatenates 1 1 with 1 1 1, the result will be identical to his concatenating 1 1 1 with 1 1" (Rotman 2000, p. 16). Following Deleuze and Guattari (1994), Rotman distinguishes ordinals and cardinals in a way that fits my thinking and seems enlightening to me: ordinals are rhythmic, directional in terms of serial continuation, and hence akin to melody; cardinals, in contrast, seem a parallel presentation, a harmony. I see this as matching the notion that ordinals are formed by focusing on the repetition of counting acts, whereas cardinals arise from reflecting a counted plurality as a unit (cf. Rotman 2000, p,146).

Much more, I am sure, has been written during the last fifteen years that would merit a comment; but my article was never intended as a review of the field.

Acknowledgment

I am indebted to David Isles, Cliff Konold, Jack Lochhead, John Richards, Les Steffe, and an anonymous reviewer for helpful comments on a draft of this paper.

Notes

1. In Newman (1956), p. 638.
2. E.g., "Mathematical objects are invented or created by humans" (Hersh 1986, pp. 22f) and particularly his reference to Piaget: "…one cannot overestimate the importance of his central in-sight: that mathematical intuitions are not absorbed from nature by passive observation, but rather are created by the experience of active manipulation of objects and symbols." (Hersh 1986, p. 26)
3. I have separated and numbered the three parts, although they form a consecutive passage in Whitehead's essay.
4. An interdisciplinary research group founded by Silvio Ceccato in the 1940s, which published the journal *Methodos* and was later incorporated as the Center for Cybernetics of Milan University.
5. Cf. Piaget's 1937 book "La construction du réel chez l'enfant" and half a dozen other volumes whose titles refer to these concepts.
6. The explanations of the term "set" one finds in textbooks (e.g. that sets are somewhat like a jury or the signs of the zodiac) do not diminish the obscurity, because they omit the crucial fact that such collections are formed on the basis of an extrinsic consideration that is quite unmathematical and that the concept of "set" requires one to take the units of the collection and deprive them of whatever attributes they might have beyond being considered units.
7. Cf. Piaget's two 1977 volumes *Recherches sur l'abstraction réfléchissante*; also Beth & Piaget (1961).
8. A full description of this pointing function can be found in Glasersfeld (1991).
9. I am avoiding the word "refer" because of its usual connotation of reference to real-world objects.
10. I might add that perplexities of this kind arose also at other times during our mathematics instruction, especially when we came to differential calculus and integration. As the perplexities mounted, more members of the class concluded that mathematics is incomprehensible and not much fun to pursue. This was a pity and could have been avoided, if only it had been made clear from the beginning that geometry is conceptual and that the world we consider external and physical does not provide geometrical entities but perceptual situations from which we may abstract them.
11. The same, incidentally, goes for "location." The visual focus of attention has no location, except relative to things one has isolated in the visual field. But this would lead to a consideration of the concept of "space," which lies beyond the scope of this discussion.
12. Ceccato later discarded this explanation because it did not reduce the concept's structure to moments of attention (Ceccato 1966, p. 500). However, as an experiential scenario that might lead to the abstraction of the concept it is still a good example.
13. To an English reader, "to draw a line" is most likely to suggest the appearance of a mark on paper or on some other surface. But Kant's emphasis shows that he wanted to draw attention to the action of the hand.
14. I was delighted to discover, on rereading Beth's introductory essay in Beth & Piaget (1961), that this idea was contained in a passage he quoted from a French philosopher of the 1920s, which I do not remember having read before:
    "To deduce is to construct. One demonstrates only hypothetical judgments; one demonstrates that one thing is the consequence of another. For that purpose one uses the hypothesis to construct the consequence. The conclusion is necessary, … They (the premises) are propositions that have been admitted beforehand, either by virtue of preceding demonstrations or as definitions or postulates." (Goblot 1922, as quoted in Beth & Piaget 1961, p. 21)
    If not explicitly hypothetical, the propositions used as premises may be inductive inferences or results of prior deductions from inductive inferences; and since inductive inferences cannot be considered logically certain, they are in this context equivalent to hypotheses. Also, it should be stressed that certainty, in this context, concerns the derivation of the conclusion from the premises and not the traditional notion of truth relative to a "real" world.
15. It seems to me that this faith in human memory is not essentially different from the confidence we are expected to have in the functioning of a computer that has carried out a "proof" that would take a human longer than a life time (cf. Tymoczko 1979/1986c).
16. About the nonverbal mental images that mathematicians make use of, Einstein said: in his case, they were "visual and some of muscular type." (Cited in Hadamard 1954, p. 143).

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Ernst von Glasersfeld was born in Munich, 1917, of Austrian parents, and grew up in Northern Italy and Switzerland. Briefly studied mathematics in Zürich and Vienna and survived the 2nd World War as farmer in Ireland. Returned to Italy in 1946, worked as journalist, and collaborated until 1961 in Ceccato's Scuola Operativa Italiana (language analysis and machine translation). From 1962 director of US-sponsored research project in computational linguistics. From 1970, he taught cognitive psychology at the University of Georgia, USA. Professor Emeritus, 1987. At present Research Associate at Scientific Reasoning Research Institute, University of Massachusetts. Dr.phil.h.c., University of Klagenfurt, 1997. – Books (among others): Wissen, Sprache und Wirklichkeit, Vieweg Verlag, Wiesbaden, 1987. Linguaggio e comunicazione nel costruttivismo radicale, CLUP, Milano, 1989. Radical Constructivism: A way of knowing and learning, Falmer Press, London, 1995 (German translation: Suhrkamp Verlag, Frankfurt am Main, 1996; also Portuguese, Korean, Italian translations.) Grenzen des Begreifens, Benteli Verlag, Bern, 1996. Wege des Wissens, Carl Auer Verlag, Heidelberg, 1997. Wie wir uns erfinden (with H. von Foerster), Heidelberg: Carl Auer, 1999. (Italian translation: Rome, Odradek, 2001). More than 260 paper publications since 1960.

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Received: 13 July 2005
Accepted: 3 October 2005